TOPOLOGY

1. Metric spaces

Definition 1.1. A *metric space* is a pair (X, d) where X is a set and d is a mapping from $X \times X$ to $\mathbb{R}$, called a *metric*, satisfying

(M1) $d(x, y) = 0$ iff $x = y$,
(M2) $d(x, y) = d(y, x)$, and
(M3) $d(x, z) \leq d(x, y) + d(y, z)$,

for all $x, y, z \in X$.

Definition 1.2. Let (X, d) be a metric space. Given a point $x \in X$, a *ball* B of radius $r > 0$ around x is the set

$$B = B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

A subset $U \subseteq X$ is *open* if for every point $x \in U$, there is a ball around x that is entirely contained in U.

Proposition 1.3. *A subset O of a metric space X is open if and only if it is a union of balls.*

Proof. Suppose O is open in X. Then for each $x \in O$ there exists a ball $B_x \subseteq O$. Consequently, $O = \bigcup_{x \in O} B_x$.

The other direction is trivial: balls are open and the arbitrary union of open sets is open. □

Proposition 1.4. *Suppose that (X, d) is a metric space and $x \neq y \in X$. Then there exists disjoint open nhoods O_x and O_y of x and y, respectively.*

Proof. Equivalently, we aim to show metric spaces are Hausdorff. Choose $O_x = B\big(x, \frac{r}{2}\big)$ and $O_y = B\big(y, \frac{r}{2}\big)$ where $r = d(x, y) > 0$, and the rest follows easily using proof by contradiction. □

Definition 1.5. Let (X, d) be a metric space. A point p is a *limit point* for a set A if every open set containing p meets $A \setminus \{p\}$. A subset C of X is *closed* iff it contains all of its limit points.

Proposition 1.6. *O is open iff $X \setminus O$ is closed, and (consequently) C is closed iff $X \setminus C$ is open.*

Proof. Suppose O is open. Clearly if x is a limit point for $X \setminus O$, then $x \notin O$. For otherwise O is an open set containing x but does not meet $X \setminus O$ (contradicting that it is a limit point for $X \setminus O$). Hence, $X \setminus O$ is closed.

Suppose C is closed, and $x \in X \setminus C$. Then x is not a limit point for C, implying that there exists some open set $U \subseteq X \setminus C$ containing x. It easily follows that $X \setminus C$ contains a ball around x and thus is open. □

Definition 1.7. Given metric spaces (X, ρ) and (Y, σ), a function $f : X \to Y$ is *continuous* if for all $x \in X$ and every $\epsilon > 0$, there exists $\delta > 0$ such that $\sigma(f(x), f(y)) < \epsilon$ whenever $\rho(x, y) < \delta$.

Theorem 1.8. *Given metric spaces (X, ρ) and (Y, σ), a function $f : X \to Y$ is continuous if and only if for every open set O in Y, $f^{-1}(O)$ is an open set in X.*

Proof. ($\Longrightarrow$) Suppose O is open in Y, and that $x \in X$. Then there exists a ball B around $f(x)$ of radius ϵ contained in O. Since f is continuous, there exists $\delta > 0$ such that $f(B(x,\delta)) \subseteq B(f(x),\epsilon) \subseteq O$. Therefore, $B(x,\delta) \subseteq f^{-1}(O)$ and so $f^{-1}(O)$ is open in X.

($\Longleftarrow$) Suppose $x \in X$ and $\epsilon > 0$. Then $B(f(x),\epsilon)$ is an open set in Y, and so $f^{-1}(B(f(x),\epsilon))$ is open in X. Necessarily there exists a ball B around x of radius δ contained in $f^{-1}(B(f(x),\epsilon))$. That is to say, $f(B(x,\delta)) \subseteq B(f(x),\epsilon)$. Thus, f is continuous. □

Definition 1.9. An *isometry* between two metric spaces (X,ρ) and (Y,σ) is a one-to-one function f from X onto Y for which $\sigma(f(x),f(y)) = \rho(x,y)$ for all $x, y \in X$. A *non-expansive* function f from X to Y is one with $\sigma(f(x),f(y)) \leq \rho(x,y)$ for all $x, y \in X$.

Note. We consider metric spaces to be equivalent if there exists an isometry between them.

Proposition 1.10. *Let (X,ρ) and (Y,σ) be metric spaces. There is an isometry between X and Y if and only if there exists non-expansive functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$ where 1_X and 1_Y are the identity functions on X and Y, respectively.*

Proof. ($\Longrightarrow$) Suppose there is an isometry $h : X \to Y$ between X and Y. Then $f = h$ and $g = h^{-1}$ has $g \circ f = 1_X$ and $f \circ g = 1_Y$. Moreover, observe that f and g are non-expansive.

($\Longleftarrow$) Suppose there exists non-expansive functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Suppose $x, y \in X$, and let $u = f(x)$ and $v = f(y)$. Then $\sigma(u,v) \leq \rho(x,y)$, and $\rho(g(u),g(v)) \leq \sigma(u,v)$. But observe that $g(u) = x$ and $g(v) = y$ such that $\sigma(f(x),f(y)) = \rho(x,y)$ for all $x, y \in X$. Obviously f is a bijection and thus is an isometry between X and Y. □

Example 1.11. For the Euclidean metric on $\mathbb{R}^2$ or $\mathbb{R}^3$, permutations and negating of entries are precisely the functions that are isometries. For example, in $\mathbb{R}^2$, $(x,y) \mapsto (y,x)$, $(x,y) \mapsto (-y,x)$ and $(x,y) \mapsto (-x,-y)$ all form isometries.

Maps of the form $x \mapsto Ax + t$ where $x, t \in \mathbb{R}^n$ and A is an orthogonal matrix $A^TA = I$ are isometries. We have translations, reflections, and rotations.

2. Topological spaces

Definition 2.1. A *topological space* is a pair $\langle X, \mathscr{T} \rangle$ where X is a set and $\mathscr{T}$ is a collection of subsets of X (called a *topology* on X) satisfying the following properties:

- $(O1)$ The empty set $\varnothing$ and X are both in $\mathscr{T}$.
- $(O2)$ The union of any collection of sets in $\mathscr{T}$ is contained in $\mathscr{T}$.
- $(O3)$ The intersection of any finite collection of sets in $\mathscr{T}$ is also contained in $\mathscr{T}$.

The elements of $\mathscr{T}$ are called *open sets*. We say C is *closed* if $X \setminus C$ is open.

Theorem 2.2. *Any non-empty open set in $\mathbb{R}$ or in $(0,1)$ is a countable disjoint union of non-empty open intervals.*

Proof. Let O be a non-empty open set in $\mathbb{R}$. Define $x \sim y$ iff $[x,y] \subseteq O$ or $[y,x] \subseteq O$ for each $x, y \in O$. $\sim$ is an equivalence relation on O. The equivalence classes are the maximal open intervals in O. Then O is a union of its equivalence classes, and these are pairwise disjoint. Each of the open intervals contain rational numbers, so there must be countably many of them. □

Example 2.3. The rationals and irrationals are neither open or closed in the usual topology on $\mathbb{R}$.

Definition 2.4. If $Y \subseteq X$ and $\mathscr{T}_X$ is a topology on X, we define the *induced subspace topology* on Y as $\mathscr{T}_Y = \{O \cap Y \mid O \in \mathscr{T}_X\}$.

Example 2.5. Consider the subspace $\mathcal{C}$ of $[0, 1]$ (with the usual topology) obtained by removing the set C_1 consisting of the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, then removing the set C_2 consisting of the (open) middle-thirds of the two resulting subintervals, and continuing in this fashion. Thus,

$$\mathcal{C} = \bigcap_{i=1}^{\infty} C_i = \left\{ x \in [0, 1] \;\middle|\; x = \sum_{n \geq 1} \frac{2x_n}{3^n}, x_n \in \{0, 1\} \right\}.$$

Note that $\mathcal{C}$ is non-empty (indeed uncountably infinite) and is an important topological space (called the *Cantor set* or *Cantor space*).

Proposition 2.6. *The Cantor space is a non-empty bounded and closed set (consequently compact).*

Proof. It is clear the Cantor space is bounded and non-empty (for example $0 \in \mathcal{C}$). Suppose x is a limit point for $\mathcal{C}$ and, for sake to derive a contradiction, further that $x \notin \mathcal{C}$. Being a limit point for $\mathcal{C}$, it follows x is a limit point for each C_i. But then $x \in C_i$ for all $i \in \mathbb{N}$, implying that $x \in \mathcal{C}$ (a clear contradiction). □

Definition 2.7. The (closed) n-cell in $\mathbb{R}^n$ is

$$E^n = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_1^2 + \ldots + x_n^2 \leq 1\right\}.$$

Its boundary is the *sphere* of dimension $n - 1$:

$$S^{n-1} = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_1^2 + \ldots + x_n^2 = 1\right\}.$$

The *open n-cell* is the interior of the closed n-cell:

$$\operatorname{Int}(E^n) := E^n \setminus S^{n-1} = \left\{(x_1, x_2, \ldots, x_n) \;\middle|\; x_1^2 + \ldots + x_n^2 < 1\right\}.$$

Definition 2.8. Let $\langle X, \mathscr{T} \rangle$ be a topological space.

- Given a point x of X, we call a subset N of X a *neighbourhood* of x if there is an open set O such that $x \in O \subseteq N$.
- The *interior* of a set A, written $\operatorname{Int}(A)$, is the union of all open sets contained within A.
- A point p of X is a *limit point* of a subset A of X if every nhood of p meets $A \setminus \{p\}$.
- The union of a subset A of X and all its limit points is called the *closure of* A and is written $\overline{A}$.

Theorem 2.9. *A set is closed if and only if it contains all of its limit points.*

Proof. ($\Longrightarrow$) Suppose C is closed in X. For sake to derive a contradiction, suppose that $x \in X \setminus C$ is a limit point of C. But $X \setminus C$ is a nhood of x and does not meet $C \setminus \{x\}$, contradicting that x is a limit point of C.

($\Longleftarrow$) Suppose C contains all of its limit points, and that $x \in X \setminus C$. Then x is not a limit point for C, therefore implying that there exists an open nhood of x disjoint with C. The union of such nhoods for every point in $X \setminus C$ is $X \setminus C$, and so $X \setminus C$ is an open set. Thus, C is closed. □

Theorem 2.10. *For any subset A of X, $\overline{A}$ is a closed subset of X. Moreover, $\overline{A}$ is the intersection of all closed subsets of X containing A (and so $\overline{A}$ is the smallest closed set containing A). Consequently, A is closed if and only if $A = \overline{A}$.*

Proof. Suppose $x \in X \setminus \overline{A}$ is a limit point of $\overline{A}$. Then x cannot be a limit point for A (for otherwise it is in $\overline{A}$). Hence, x must be a limit point of $\overline{A} \setminus A$, and so every nhood of x meets $\overline{A} \setminus A$. Moreover, there must exist an open nhood N of x disjoint from A (since x is not a limit point of A). But N meets $\overline{A} \setminus A$, that is, N contains a limit point y of A. Therefore, N is an open nhood of y and so must meet A (a contradiction). Thus, $\overline{A}$ contains all of its limit points and is consequently closed. □

Proposition 2.11. *The function $A \mapsto \overline{A}$ satisfies the following four properties of a topological closure operator:*

(1) Extensive*: $A \subseteq \overline{A}$,*
(2) Monotone*: $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$,*
(3) Idempotent*: $\overline{\overline{A}} = \overline{A}$,*
(4) Topological*: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

Proof. By definition, both (1) and (2) are clear. As for (3), this is proven by the result above. Lastly, (4) is satisfied by observing

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$$

and

$$A \cup B \subseteq \overline{A} \cup \overline{B} \Longrightarrow \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

□

Definition 2.12. A subset Y of X is *dense* if its closure equals X.

Definition 2.13. A *base* for a topological space X is a collection $\mathscr{B}$ of open sets in X such that every open set in X is the union of members of $\mathscr{B}$. Equivalently, for every $x \in X$ and every nhood N of x, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq N$.

Theorem 2.14. *Let $\mathscr{B}$ be a non-empty collection of subsets of a set X. If the intersection of any finite number of members of $\mathscr{B}$ is contained in $\mathscr{B}$ and $\bigcup \mathscr{B} = X$, then $\mathscr{B}$ is a base for a topology on X.*

Proposition 2.15. *If D is dense in a metric space, then the set of all open balls with centres in D and rational radii is a base for the associated topology.*

Proof. Let $x \in X$ and N be a nhood of x. Then there is some open set U such that $x \in U \subseteq N$. Moreover, there exists a ball $B(x, r) \subseteq U$. Since D is dense, there is a $y \in D \cap B\big(x, \frac{r}{2}\big)$. Now, $x \in B\big(y, \frac{r}{2}\big) \subseteq B(x, r) \subseteq N$. □

3. Continuous functions, homeomorphisms and embeddings

Definition 3.1. Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff the pre-image of every open set in Y is open in X. We refer to a continuous function simply as a *map*.

Lemma 3.2. (1) *A function $f : X \to Y$ is continuous iff the pre-image of every closed set in Y is a closed set in X.*
(2) *A function $f : X \to Y$ is continuous if for all $x \in X$ and any nhood V of $f(x)$ there is a nhood U of x such that $f(U) \subseteq V$.*
(3) *A composition of two continuous functions is continuous.*

Proof. Suppose f is continuous, and that C is closed in Y. Then $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is closed in X. Hence, the pre-image of C is a closed set in X. The converse follows as easily, proving (1).

Suppose for all $x \in X$ and any nhood V of $f(x)$ there is a nhood U of x such that $f(U) \subseteq V$. Suppose V is open in Y. Then for all $x \in f^{-1}(V)$ there is a nhood U_x

of x such that $f(U_x) \subseteq V$. Because $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$, we get that $f^{-1}(V)$ is open such that f is continuous, proving (2).

Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Since $(g \circ f)^{-1}(U) = f^{-1}\big(g^{-1}(U)\big)$ for any open set U in Z, it is clear the pre-image of U w.r.t. $g \circ f$ is open in X. □

Theorem 3.3. (Space filling curve) *There is a continuous function $f : E^1 \to E^2$ that is onto. Moreover, for any $n > 1$, there is a continuous function from E^1 onto E^n and a continuous function from $(0, 1)$ onto $\mathbb{R}^n$.*

Lemma 3.4. (Glueing lemma) *Suppose that $X = A \cup B$ where A and B are closed. Let $f : X \to Y$ be a function with $\alpha = f\mid_A$ and $\beta = f\mid_B$ being both continuous. Then f is continuous.*

Proof. Suppose C is closed in Y. Then $\alpha^{-1}(C)$ and $\beta^{-1}(C)$ are closed in their respective subspace topologies on X. Observe that $\alpha^{-1}(C) = f^{-1}(C) \cap A$ and $\beta^{-1}(C) = f^{-1}(C) \cap B$. Since A and B are closed in X, we must have that $\alpha^{-1}(C)$ and $\beta^{-1}(C)$ are closed in X. Thus, $\alpha^{-1}(C) \cup \beta^{-1}(C) = f^{-1}(C)$ is closed in X. □

Definition 3.5. Given two topological spaces X and Y, a function $f : X \to Y$ is said to be a *homeomorphism* from X to Y if f is a bijection that is continuous, and f^{-1} is also continuous. When such a function exists, we say the topological spaces are *homeomorphic*, written $X \cong Y$. If there is a heomomorphism h from X to a subspace of Y, then we say h is an *embedding* of X into Y.

Lemma 3.6. *A continuous map f from a topological space X to another topological space Y is a homeomorphism iff there is a continuous map g from Y to X so that $g \circ f = 1_X$ and $f \circ g = 1_Y$ where 1_X and 1_Y are the identity functions on X and Y, respectively.*

Proof. The proof is trivial, simply take $g = f^{-1}$. □

Example 3.7. The open interval $(0, 1)$ is homeomorphic to $\mathbb{R}^{>0}$ and to $\mathbb{R}$. We have that $f : (0, 1) \to \mathbb{R}^{>0}$ defined $f(x) = -\ln(x)$ is a homeomorphism.

Example 3.8. S^1 is homeomorphic to the boundary of a square. Project points from circle onto the square, 'radial projection' from the centre of the sphere and square yields a homeomorphism.

Example 3.9. The open interval and the circle minus a point $S^1 \setminus \{p\}$ are homeomorphic. Take $f : (0, 1) \to S^1 \setminus \{p\}$ defined $f(t) = pe^{2i\pi t}$ (stereographic projection).

Consider the half-open interval $(0, 1]$ and circle S^1. There is a one-to-one continuous map f from $(0, 1]$ onto S^1, but its inverse is not continuous (these two spaces are not homeomorphic). Extend the map f above.

Example 3.10. $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$ for all $n \geq 1$. For $n = 2$, a homeomorphism is provided by stereographic projection of $S^2 \setminus \{p\}$ from the point p onto $\mathbb{R}^2$. In coordinates, $p = (0, 0, 1)$ project onto xy-plane via $f(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

Example 3.11. The closed n-cell E^n is homeomorphic to $[0, 1]^n$ and to the n-*simplex* Δ^n defined by:

$$\Delta^n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \ldots + x_n \leq 1, x_i \geq 0\}.$$

Example 3.12. The inclusion map $x \mapsto x$ from S^2 into $\mathbb{R}^3$ is an embedding of S^2 into $\mathbb{R}^3$.

$S^1 \times S^1 = \left\{(u, v, x, y) \in \mathbb{R}^4 \;\middle|\; u^2 + v^2 = x^2 + y^2 = 1\right\}$ embeds into $\mathbb{R}^3$ via the map $h(u, v, , y) = ((2 + u)x, (2 + u)y, v)$.

Note. There is no embedding of S^2 into $\mathbb{R}^2$. The Klein bottle cannot be embedded in $\mathbb{R}^3$ but can be in $\mathbb{R}^4$.

Example 3.13. Let X be the set of all countably infinite binary sequences:

$$X = \{x = (x_1, x_2, \ldots) \mid x_i \in \{0,1\}\}.$$

For a finite subset S of $\{1, 2, \ldots\}$ and a function $f : S \to \{0,1\}$ let $X_{S,f}$ be the subset of X defined by

$$X_{S,f} = \{x \in X \mid x_i = f(i) \text{ for all } i \in S\}.$$

A subset B of X is called a *cylinder set* in X if $B = X_{S,f}$ for some pair S, f. The collection of cylinder sets forms a base for a topology on X, and the resulting topological space $\mathcal{C}^*$.

Note. $X_{S,f} \cap X_{S',f'} = X_{S \cup S', f \cup f'}$ where

$$f \cup f' = \begin{cases} f(i) & \text{if } i \in S; \\ f'(i) & \text{if } i \in S'; \end{cases}$$

iff f and f' agree on $S \cap S'$. Otherwise, their intersection is empty.

Proposition 3.14. *Cylinder sets are closed in* $\mathcal{C}^*$.

Proof. Let $B = X_{S,f}$ be a cylinder set in $\mathcal{C}^*$. Then

$$\begin{aligned} \mathcal{C}^* \setminus B &= \{x \in \mathcal{C}^* \mid x_i \neq f(i) \text{ for some } i \in S\} \\ &= \bigcup_{i \in S} X_{\{i\}, g_i} \end{aligned}$$

where $g_i(i) = 1 - f(i)$. □

Proposition 3.15. *Singleton sets are closed in* $\mathcal{C}^*$.

Proof. Fix $x \in \mathcal{C}^*$. For each $i \in \mathbb{N}$, let $U_i = X_{S_i, f_i}$ where $S_i = \{i\}$ and $f_i : \{i\} \to \{0,1\}$ is defined $f_i(i) = 1 - x_i$. Then $x \notin U_i$ for all i, each U_i is open, and $\bigcup_{i \in \mathbb{N}} U_i = \mathcal{C}^* \setminus \{x\}$ is therefore open. Thus, singleton sets are closed. □

Example 3.16. The topological space $\mathcal{C}^*$ is homeomorphic to the Cantor space $\mathcal{C}$. Specifically, the following map $f : C^* \to [0,1]$ with image the Cantor set $\mathcal{C}$. For $x = (x_1, x_2, \ldots) \in X$ (where $x_i \in \{0,1\}$), define

$$f(x) = \sum_{n \geq 1} \frac{2x_n}{3}.$$

Proof. Clearly, f is a bijection from X to $\mathcal{C}$. Now, f is continuous: Let $x \in X$ and $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\sum_{k \geq n} \frac{2}{3^k} < \epsilon$. Define $S = \{1, \ldots, n\}$ and $f : S \to \{0,1\}$ where $f : i \mapsto x_i$. $X_{S,f}$ consists of all sequences that have the same first n terms as x. $y \in X_{S,f}$ has

$$\begin{aligned} |f(y) - f(x)| &= \left| \sum_{k > n} \frac{2(y_k - x_k)}{3} \right| \\ &\leq \sum_{k \geq n} \frac{2}{3^k} < \epsilon. \end{aligned}$$

Thus, f is continuous at x. □

4. Huasdorff, connected and compact spaces

Definition 4.1. A space is *Hausdorff* if for any two points $x \neq y \in X$, there exist disjoint open nhoods U and V of x and y, respectively.

Proposition 4.2. *If X is Hausdorff and $x \in X$, then $\{x\}$ is closed.*

Proof. Suppose X is Hausdorff, and $x \in X$. Then given any $y \neq x \in X$, there exist open nhood U_y of y with $x \notin U_y$. Consequently, $\bigcup_{y \in X \setminus \{x\}} U_y = X \setminus \{x\}$ is the arbitrary union of open sets and is therefore open. Thus, $\{x\}$ is closed. □

Theorem 4.3. *Suppose f and g are continuous functions from a space X into a Hausdorff space Y and suppose that images of f and g on a subset U of X are equal. If U is dense in X, then $f = g$.*

Proof. Suppose that $f(x) \neq g(x)$ for some $x \in X \setminus U$. Then there exist disjoint open nhoods V_f and V_g of $f(x)$ and $g(x)$, respectively. By continuity, there exist open nhoods U_f and U_g of x with $f(U_f) \subseteq V_f$ and $g(U_g) \subseteq V_g$. Since U is dense, we must have that there exists some $y \in U$ with $y \in U_f \cap U_g$. But then $f(y) \in V_f$ and $g(y) \in V_g$ where $f(y) = g(y)$, contradicting that V_f and V_g are disjoint. Thus, $f = g$. □

Corollary 4.4. *Suppose f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(\mathbb{Q}) = g(\mathbb{Q})$. Then $f = g$.*

Definition 4.5. A topological space X is *disconnected* if there are disjoint non-empty open sets A and B such that $X = A \cup B$. Otherwise, X is *connected.* Equivalently, X is connected if the only clopen sets are $\varnothing$ and X.

If X is disconnected, then the maximal subsets Y of X for which Y is connected (under the induced subspace topology) are referred to as the *connected components* of X. Note that the connected components partition X.

Lemma 4.6. *X is connected if and only if for every pair of non-empty subsets A, B of X having union X, we have $\overline{A} \cap B \neq \varnothing$ or $A \cap \overline{B} \neq \varnothing$.*

Proof. ($\Longrightarrow$) Suppose X is a connected space, and that A, B are non-empty subsets of X with union X. If $\overline{A} \cap B = \varnothing$, then we must have that $\overline{A} = A$. Similarly, if $A \cap \overline{B}$ then $\overline{B} = B$. It follows that both cannot hold, since X would be otherwise be disconnected.

($\Longleftarrow$) Suppose for every pair of non-empty subsets A, B of X having union X, we have $\overline{A} \cap B \neq \varnothing$ or $A \cap \overline{B} \neq \varnothing$. If X were disconnected, then there are disjoint non-empty open sets U and V such that $X = U \cup V$. But then $A = X \setminus U$ and $B = X \setminus V$ are disjoint closed sets, contradicting our assumption. □

Theorem 4.7. *The real line $\mathbb{R}$ (with the usual topology) is connected, and the connected subsets of $\mathbb{R}$ are the intervals.*

Proof. It suffices to show that $\mathbb{R}$ is connected. Then all other intervals are connected. Suppose that $\mathbb{R}$ is disconnected. Then there is an open partition of $\mathbb{R}$: $\mathbb{R} = O_1 \cup O_2$, where O_1 and O_2 are disjoint non-empty open sets. O_1 and O_2 are the disjoint union of open intervals, and so is $\mathbb{R}$. Let (a, b) be one of these intervals in the union. Consider $b \in \mathbb{R}$. It belongs to an open interval in the union, say (c, d). But then $(a, b) \cap (c, d) \neq \varnothing$, a contradiction.

Suppose we have $\varnothing \neq C \subseteq \mathbb{R}$ is connected but C is not an interval. Then there are $a < c < b$ with $a, b \in C$, $c \notin C$. Let $O_1 = C \cap (-\infty, c)$ and $O_2 = C \cap (c, \infty)$. Then O_1, O_2 are open in C. Moreover, $a \in O_1$, $b \in O_2$ and $O_1 \cap O_2 = \varnothing$ where $O_1 \cup O_2 = C$. So, C is not connected, a contradiction. □

Theorem 4.8. *If X is a connected topological space and if $f : X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. Suppose X is a connected topological space and $f : X \to Y$ is continuous. For sake to derive a contradiction, suppose $f(X) = A \cup B$ where A, B are disjoint open sets. By continuity of f, both $f^{-1}(A)$ and $f^{-1}(B)$ are open in X. But then $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A), f^{-1}(B)$ are disjoint open sets, contradicting X is connected. Thus, $f(X)$ is connected. □

Corollary 4.9. (Intermediate value theorem) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0$ and $f(b) > 0$. Then $f(x) = 0$ for at least one $x \in (a, b)$.*

Proof. $[a, b]$ is connected and if f is continuous, the image of f is connected. Therefore, the image of f is an interval. $f(a) < 0 < f(b)$ implies $0 \in f([a, b])$. Thus, $f(x) = 0$ for at least one $x \in (a, b)$. □

Definition 4.10. A topological space X is *path-connected* if for any two points $x, y \in X$ there exists a path from x to y in X, that is, a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$.

Example 4.11. $\mathbb{R}^n$ and S^n, $n \geq 1$, are path-connected. $\mathbb{R}^2 \setminus \{p\}$ for any $p \in \mathbb{R}^2$ is path-connected.

Theorem 4.12. *A path-connected space is connected.*

Proof. Suppose X is a path-connected space but disconnected. Then there exists disjoint non-empty open sets A, B such that $X = A \cup B$. Let $x \in A$ and $y \in B$ be arbitrary. Then there exists a continuous function $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. But $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty open sets with their union $[0, 1]$. This contradictions $[0, 1]$ being connected. □

Corollary 4.13. $\mathbb{R}$ *and* $\mathbb{R}^2$ *are not homeomorphic.*

Proof. Assume there is a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}$. Let $p \in \mathbb{R}^2$. Then the restriction $f \mid \mathbb{R}^2 \setminus \{p\}$ is also a homeomorphism. But $\mathbb{R}^2 \setminus \{p\}$ is connected and $\mathbb{R} \setminus \{f(p)\}$ is disconnected, a contradiction. □

Theorem 4.14. *A connected open subset of Euclidean space is path-connected.*

Proof. Let X be a non-empty connected open subspace of $\mathbb{R}^n$. Let $x \in X$ and

$$U(x) = \{y \in X \mid y \text{ can be joined to } x \text{ by a path}\}.$$

X is partitioned into sets of the form $U(x)$ for suitable $x \in X$. If $U(x) \cap U(y) \neq \varnothing$, then $U(x) = U(y)$. Claim: $U(x)$ is open in X. If this is true, then by connectedness of X, $U(x) = X$, and so X is path-connected. Let $y \in U(x)$. Now, X is an open nhood of y. So there is a $r > 0$ such that $B(y, r) \subseteq X$. Any point in $B(y, r)$ can be joined to y by a path, and so $B(y, r) \subseteq U(x)$. It follows that $U(x)$ is open, as claimed. □

Example 4.15. The topologists sine curve

$$T = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \,\middle|\, x \in (0, 1] \right\} \cup \{(0, 0)\}$$

is connected but not path-connected.

Proof. Suppose $f : [0, 1] \to T$ is continuous and $f(0) = (0, 0)$. Suppose T is disconnected, $T = U \cup V$ where U, V are disjoint non-empty open sets. Without loss of generality, $(0, 0) \in U$. Then $U = W \cap T$ where $W \subseteq \mathbb{R}^2$ is open. Therefore W contains an open ball $B = B((0, 0), r)$ for some $r > 0$. Then $B \cap T$ contains points not equal to $(0, 0)$. Now, $U \setminus \{(0, 0)\}$ is non-empty, and $T \setminus \{(0, 0)\} =$

$(U \setminus \{(0,0)\}) \cup V$ is the disjoint union of two non-empty open sets. But $T \setminus \{(0,0)\}$ is a graph of $x \mapsto \sin\left(\frac{1}{x}\right)$ which is connected, a contradiction. □

Definition 4.16. A topological space X is said to be *locally path-connected* if for each $x \in X$ and each nhood U of x there is a path-connected nhood V of x that is contained in U.

Example 4.17. $\mathbb{R}^n$ is locally path connected. Note that locally path connected spaces need not be path-connected or even connected (take two open balls that are disjoint).

A path-connected space need not be locally path-connected. For example, consider $X = T \cup L$ where

$$L = \{(0,y) \mid y \in [0,1]\} \cup \{(x,1) \mid x \in [0,1]\}.$$

Then X is path-connected, L is path-connected, $T \setminus \{(0,0)\}$ is path-connected and $L \cap T \setminus \{(0,0)\} \neq \varnothing$.

Proposition 4.18. *A connected, locally path-connected topological space is path-connected.*

Theorem 4.19. (Jordan Curve Theorem). *Suppose that a subset C of $\mathbb{R}^2$ is homeomorphic to S^1. Then $\mathbb{R}^2 \setminus C$ has exactly two connected components, one bounded (the 'inside') and the other unbounded (the 'outside').*

Note. Each of the two components have C as their boundary, and the 'inside' and the 'outside' planar region determined by C are homeomorphic to the interior and exterior of the unit disk (Jordan-Schonflies theorem).

Definition 4.20. Given a subset A of a topological space X, an *open cover* of A is a collection $\mathscr{A}$ of open sets in X whose union contains A. A *finite subcover* of $\mathscr{A}$ is a finite subcollection $\mathscr{B}$ of $\mathscr{A}$ whose union contains A. A subset A of a topological space X is *compact* if every open cover of A has a finite subcover.

Example 4.21. $[0,1)$ and $[0,\infty)$ are not compact. $U_n = [0, 1-\frac{1}{n})$ is open in $[0,1)$, and covers $[0,1)$. But it has no finite subcover. $U_n = [0,n)$ is open in $[0,\infty)$, and similarly covers, but has no finite subcover, $[0,\infty)$.

The interval $[a,b]$ is compact (the Heine-Borel theorem in dimension 1). Let $\mathscr{U}$ be an open cover of $[a,b]$. Let $F = \{x \in [a,b] \mid [a,x]$ is finitely covered by $\mathscr{U}\}$. Let $s = \sup F$. Since F is bounded, the supremum s exists. $s \in [a,b]$ and so $s \in U$ for some $U \in \mathscr{U}$. There is a $x_s \in F$ such that $x_s \in U$. $[a, x_s]$ is finitely covered and so is $[a,s]$. Thus, $s \in F$. Claim: $s = b$. If $s \neq b$, then there exists some $c \in U$ such that $c > s$, but by similar argument $c \in F$: contradicting $\sup F = s$.

Theorem 4.22. *A closed subspace of a compact space is compact.*

Proof. Suppose C is a closed subspace of a compact space X. If $\mathscr{U}$ is an open cover for C, then $\mathscr{U} \cup \{X \setminus C\}$ is an open cover for X. By compactness, there exists finite subcover $\mathscr{V}$ for X. Thus, $\mathscr{V} \setminus \{X \setminus C\}$ is a finite subcover of $\mathscr{U}$ for C and so C is compact. □

Theorem 4.23. *A compact subspace of a Hausdorff space is closed.*

Proof. Suppose K is a compact subspace of a Hausdorff space X. Fix $y \in X \setminus K$. Then given any $x \in K$, we have that there exists disjoint open nhoods U_x and V_x of x and y, respectively. Consequently, $\mathscr{A} = \{U_x \mid x \in K\}$ is an open cover for K. By compactness, there exists finite subcover $\mathscr{B}$ for K. Moreover, for each $U_x \in \mathscr{B}$ we consider corresponding V_x and denote their intersection V_y. Then V_y is an open nhood of y disjoint from K. Thus, $\bigcup_{y \in X \setminus K} V_y = X \setminus K$ is an open set, and so K is closed. □

Theorem 4.24. *In a metric space, any compact subspace is closed and bounded.*

Proof. Metric spaces are Hausdorff and so it is clear that compact subspaces are closed. Now, suppose K is a compact subspace of a metric space (M, ρ). For sake to derive a contradiction, suppose that K is not bounded. Then there exists $x \in M$ such that given any $\epsilon > 0$ there exists $y \in K$ such that $\rho(x, y) > \epsilon$. Construct the sequence $\{x_n\}$ such that $x_n \in K$ and $\rho(x_n, x) > \frac{1}{n}$ for each n. But it is clear that no convergent subsequence exists, contradicting K is compact. Thus, K is bounded. □

Note. Suppose K is a compact subspace of a metric space M. Then K is closed because M is Hausdorff. If $x \in K$, the balls $B(x, n)$ form an open cover for A. Since A is compact, there are finitely many and so $A \subseteq B(x, r)$ for some $r > 0$.

Example 4.25. In a metric space there may exist a closed and bounded set that is not compact. Take any discrete metric space on an infinite set.

Theorem 4.26. (Heine-Borel theorem). *A subset A of $\mathbb{R}^n$ is compact if and only if A is closed and bounded.*

Proof. $A \subseteq \mathbb{R}^n$ compact implies A is closed and bounded by the above.

Conversely, $n = 1$ already seen in examples, and $n > 1$ uses product spaces. □

Theorem 4.27. *If $f : X \to Y$ is continuous and $K \subseteq X$ is compact, then $f(K)$ is compact.*

Proof. Suppose $\mathscr{U}$ is an open cover for $f(K)$. Then $\{f^{-1}(U) \mid U \in \mathscr{U}\}$ is an open cover for K. Consequently, there exists finite subcover $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$ for K. Then $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathscr{U}$ for $f(K)$. Thus, $f(K)$ is compact. □

Example 4.28. S^2 not homeomorphic to $\mathbb{R}^2$ because S^2 is compact but $\mathbb{R}^2$ is not.

Corollary 4.29. (Extreme value theorem). *Suppose that K is a compact space and $f : K \to \mathbb{R}$ is continuous. Then f is bounded and there exist $p, q \in K$ such that $f(p) = \sum_{x \in K} f(x)$ and $f(q) = \inf_{x \in K} f(x)$.*

Proof. $f(K)$ is compact and so it is closed and bounded. Hence, $f(K) = [a, b]$ for some $a, b \in \mathbb{R}$ and the result easily follows. □

Theorem 4.30. *An infinite subset of a compact space must have a limit point.*

Proof. Suppose A is an infinite subset of a compact space X. For sake to derive a contradiction, suppose A has no limit points. Given any $p \in X$, there exists open nhood U_p of p such that U_p and $A \setminus \{p\}$ are disjoint. Then $\mathscr{U} = \{U_p \mid p \in X\}$ is an open cover for X. Hence, there exists finite subcover $\mathscr{V}$ for X. But $\bigcup \mathscr{V} \neq X$ because it only shares finitely points with A, a contradiction. □

Theorem 4.31. *Suppose that f is a continuous one-to-one function from a compact space X onto a Hausdorff space Y. Then f is a homeomorphism.*

Proof. Suppose f is a continuous one-to-one function from a compact space X onto a Hausdorff space Y. Suppose C is closed in X. Then C is compact. Therefore, $f(C)$ is compact. Since Y is Hausdorff, we get that $f(C)$ is a closed set. Thus, f is a closed map and therefore a homeomorphism. □

Theorem 4.32. (Heine-Cantor theorem). *If M, N are metric spaces, $f : M \to N$ is a continuous function and M is compact, then f is uniformly continuous.*

Theorem 4.33. *Suppose X and Y are homeomorphic. In this case, X is Hausdorff iff Y is Hausdorff, X is connected iff Y is connected, and X is compact iff Y is compact.*

Note. Compactness of X is equivalent to the following property: If $\mathscr{C}$ is any collection of closed sets in X and the intersection of every finite subcollection of sets in $\mathscr{C}$ is non-empty, then the intersection of all the sets in $\mathscr{C}$ is non-empty.

Definition 4.34. A topological space X is *normal* iff for any two disjoint closed subsets C, F of X, there exists disjoint open sets U and V of X with $C \subseteq U$ and $F \subseteq V$.

Note. The *Tietze extension theorem* states that if X is a normal space and C is any closed subset of X and f is a continuous function from C into $\mathbb{R}$ (with its usual topology) then there is a continuous function $F : X \to \mathbb{R}$ that extends f; moreover, the supremum and infimum of values of F and f agree.

The *Urysohn metrisation theorem* states that if X is a normal topological space with a countable basis, then there is a metric d on X such that the open sets with respect to the metric d are exactly the open sets of X.

A surprising result is the following: A topological space Y is a compact metric space if and only if Y is the continuous image of the Cantor space X_C. That is, there is an onto continuous function $f : X_C \to Y$.

We can now provide a short non-constructive proof of the existance of a space-filling curve (i.e., continuous f from $[0, 1]$ onto $[0, 1]^2$). View the Cantor space X_C has a closed subset of $[0, 1]$. Now, the unit square $[0, 1]^2$ is a compact metric space so there is a continuous function f from X_C onto $[0, 1]^2$. Then extend f to $[0, 1]$ by the Tietze extension theorem (since X_C is closed, $[0, 1]$ normal) applied to f_1 and f_2, where $f(x) = (f_1(x), f_2(x))$. Alternatively, one can extend f by piecewise linear functions on the removed open intervals in the 'middle-thirds' construction of X_C.

Proposition 4.35. *Metric spaces and compact Hausdorff spaces are normal.*

Proof. Suppose X is a compact Hausdorff space, and C, F are disjoint closed sets in X. Then X is regular, so for each $x \in F$ there exists disjoint open sets U_x and V_x containing x and C. Since F is compact, we have that there are finitely many $U_{x_1}, \ldots, U_{x_n}$ which cover F. Let $U = \bigcup_{i=1}^n U_{x_i}$ and $V = \bigcap_{i=1}^n V_{x_i}$. Then U and V are non-empty disjoint open sets containing C and F, respectively. □

Definition 4.36. A topological space X is said to be *locally compact* if each $x \in X$ has a compact nhood.

Example 4.37. Every compact space is locally compact; $\mathbb{R}^n$ is locally compact but not compact; the (open) cylinder $S^1 \times \mathbb{R}$ in $\mathbb{R}^3$ is locally compact but not compact; every infinite discrete space is locally compact but not compact.

Definition 4.38. A topological space X is said to be *regular* iff for every closed set C and $x \notin C$, there exist disjoint open sets U and V containing x and C, respectively.

Proposition 4.39. *Every locally compact Hausdorff space is regular.*

Proof. Let X is a locally compact Hausdorff space, and C be closed in X. Suppose $x \notin C$. There exists a compact nhood W of x. Since C is closed, we have that $C \cap W$ is compact in the subspace topology W of X. For each $y \in C \cap W$, by the Hausdorff property there exists disjoint open nhoods U_y and V_y of x and y, respectively. As V_y covers $C \cap W$, we can take finitely many $V_1, \ldots, V_n$ to cover $C \cap W$. Let V be the union of such V_i. For the corresponding U_i, let $U = \bigcap_{i=1}^n U_i \cap \operatorname{Int}(W)$. Then

U is an open nhood of x disjoint with V, where V contains $C \cap W$. Furthermore, as X is Hausdorff and W is compact, $X \setminus W$ is open and contains no point of U. Thus, U and $(X \setminus W) \cup V$ are disjoint open nhoods of x and C, respectively, giving us that X is regular. $\square$

Proposition 4.40. *A topological space X is regular if and only if for each open set U in X and $x \in U$, there is an open set V containing x such that $\overline{V} \subseteq U$.*

Proof. ($\Longrightarrow$) Suppose X is regular, and $x \in U$ where U is open in X. Since X is regular, there are disjoint open sets V and W containing x and $X \setminus U$, respectively. Hence, $\overline{V} \subseteq X \setminus W \subseteq U$.

($\Longleftarrow$) Assume for each open set U in X and $x \in U$, there is an open set V containing x such that $\overline{V} \subseteq U$. Suppose C is closed in X and $x \notin C$. Then there is an open set V containing x such that $\overline{V} \subseteq X \setminus C$. That is to say, $C \subseteq X \setminus \overline{V}$. Thus, V and $X \setminus \overline{V}$ are disjoint open sets containing x and C, respectively; it follows X is regular. $\square$

Proposition 4.41. *A Hausdorff space X is locally compact iff for each $x \in X$ and each nhood U of x there is a compact nhood V of x that is contained in U.*

Proof. ($\Longrightarrow$) Suppose X is locally compact, $x \in X$ and U is an open nhood of x. Being locally compact, there exists a compact nhood K of x. Let $V = \operatorname{Int}(U \cap K)$, which is an open nhood of x contained in U. Now, $\overline{V}^X$ is compact and Hausdorff, so it is regular. Since V is a nhood of x in $\overline{V}^X$, there exists open nhood W of x in $\overline{V}^X$ with $\overline{W}^{\overline{V}^X} \subseteq V$. Now, W is open in V and hence in X, and $\overline{W}^{\overline{V}^X}$ is closed in $\overline{V}^X$ and hence compact. Thus, $\overline{W}^{\overline{V}^X}$ is a compact nhood of x contained in U.

($\Longleftarrow$) Suppose for each $x \in X$ and each nhood U of x there is a compact nhood V of x that is contained in U. Obviously this means each $x \in X$ has a compact nhood and so X is locally compact. $\square$

Theorem 4.42. *Every non-empty open or closed subset of a locally compact Hausdorff space is locally compact in the induced topology.*

Proof. Suppose $Y \subseteq X$ is closed and $p \in Y$. Then p has a compact nhood W in X, and $Y \cap W$ is a compact nhood of p in Y.

Suppose $Y \subseteq X$ is open and $p \in Y$. Then Y is a nhood of p in X, so there is a compact nhood of p contained in Y. $\square$

Theorem 4.43. *A subspace Y of a locally compact Hausdorff space X is locally compact if and only if $Y = O \cap C$, where O is open and C is closed in X.*

Proof. ($\Longrightarrow$) Suppose the subspace Y of a locally compact Hausdorff space X is locally compact. It suffices to show that Y is open in $\overline{Y}$. Suppose $x \in Y$. Then there exists an open nhood U of x in Y where $\overline{U}^Y = \overline{U} \cap Y$ is compact and in particular closed in $\overline{Y}$. But $U \subseteq \overline{U} \cap Y$, so $\overline{U} \subseteq \overline{U} \cap Y \subseteq Y$. Let W be open in $\overline{Y}$ such that $W \cap Y = U$. Then

$$x \in W \subseteq \overline{W} = \overline{W \cap Y} = \overline{U} \subseteq Y.$$

($\Longleftarrow$) Suppose $Y = O \cap C$, where O is open and C is closed in X. Then Y is open in the relative topology C of X. Being a closed set, it follows by the above that C is locally compact. Thus, Y is locally compact. $\square$

5. Product and quotient spaces

5.1. **Product spaces.**

Definition 5.1. Let $\langle X, \mathscr{T}\rangle$ and $\langle X, \mathscr{T}'\rangle$ be topological spaces. The *product space* of X and Y is the topological space $X \times Y$ and has a base with sets of form $O \times O'$ where $O \in \mathscr{T}$ and $O' \in \mathscr{T}'$.

Proposition 5.2. *If $\mathscr{B}$ is a base for X and $\mathscr{B}'$ is a base for Y, then a base for $X \times Y$ is the collection of set of the form $O \times O'$ where $O \in \mathscr{B}$ and $O' \in \mathscr{B}'$.*

Proof. $O = (O_1 \times O_1') \cap (O_2 \times O_2') = (O_1 \cap O_2) \times (O_1' \cap O_2')$, where $O_1 \cap O_2$ is open in X and $O_1' \cap O_2'$ is open in Y. Hence, O is in base or empty. □

Proposition 5.3. *N is a nhood of (x, y) in $X \times Y$ iff there are open nhoods O_x and O_y of x and y, respectively, such that $O_x \times O_y \subseteq N$.*

Proof. Sufficiency is immediate, so we prove necessity; suppose N is a nhood of (x, y) in $X \times Y$. Then there is an open nhood O of (x, y) contained in N. Hence, $(x, y) \in O_x \times O_y \subseteq O \subseteq N$ for some open nhoods O_x and O_y of x and y, respectively. □

Example 5.4. $\mathbb{R} \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^2$.

$[0, 1] \times [0, 1]$ is the unit square which is homeomorphic to the unit disc. The product $(0, 1) \times (0, 1)$ is homeomorphic to $\mathbb{R}^2$.

$S^1 \times [0, 1]$ is a cylinder, and $S^1 \times S^1$ is a torus.

Note. Notice that for two topological spaces X_1 and X_2 we have two (continuous onto) *projection* maps $p_j : X_1 \times X_2 \to X_j$ $(j = 1, 2)$. It is easy to show that p_j maps open sets to open sets (this is clear for the base open sets, and each open set in $X_1 \times X_2$ is a union of base sets).

The notion of a product extends naturally to any finite number of spaces, and products preserve the properties of spaces we have considered as described in the following theorem.

Theorem 5.5.
(1) (Tychonoff's theorem). *A product of two (or more) compact spaces is compact.*
(2) *A product of two (or more) Hausdorff spaces is Hausdorff.*
(3) *A product of two (or more) connected spaces is connected.*

Proof. Let X, Y be compact, and $\mathscr{A}$ be an open cover of $X \times Y$. Call $A \subseteq X$ *good* if $A \times Y$ is covered by a finite subcover of $\mathscr{A}$. For each $x \in X$ there is an open nhood which is good. Fix $x \in X$. Each (x, y) is contained in some element of $\mathscr{A}$, say $A(x, y)$. So there are open nhoods $U(y)$ of x and $V(y)$ of y such that $U(y) \times V(y) \subseteq A(x, y)$. Now, the $V(y)$ form an open cover of Y. Since Y is compact, there are finitely many $V(y_1), \ldots, V(y_n)$ that cover Y. Define $O_x = \bigcap_{i=1}^n U(y_i)$, which is an open nhood of x. Hence, O_x is good:

$$O_x \times Y \subseteq A(x, y_1) \cup \ldots \cup A(x, y_n).$$

Finally, the O_x for each $x \in X$ is an open cover for X. By compactness of X, there are finitely many that cover X:

$$X = O(x_1) \cup \ldots \cup O(x_m),$$

where each $O(x_i)$ is good. Thus, X is good. □

Note. In fact, the converse of these results hold. We can now finish the proof of the non-trivial direction of the Heine-Borel theorem which stated that: a subset A of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Proof. $X \subseteq \mathbb{R}^n$ closed and bounded. Then $X \subseteq [a_1, b_1] \times \ldots \times [a_n, b_n]$ where each $[a_i, b_i]$ are compact, implying $[a_1, b_1] \times \ldots \times [a_n, b_n]$ is compact by Tychonoff's theorem. Now, X is a closed subspace of a compact space and so is also compact. □

Example 5.6. Note that if $X \times Y$ is homeomorphic to $X' \times Y$, this does not imply that X and X' are homeomorphic. An example is by taking $X = [0, 1]$ and $X' = Y = [0, 1)$.

5.2. Quotient spaces.

Definition 5.7. Let $\sim$ be an equivalence relation on a topological space X. The set of equivalence classes $X/_\sim$ becomes a topological space by declaring a subset O of $X/_\sim$ to be open if and only if $p^{-1}(O)$ is an open set in X. Explicitly,

$$X/_\sim = \{O \subseteq X \mid p^{-1}(O) \text{ is open in } X\}.$$

We call this space the *quotient space* of X relative $\sim$, and the associated projection map $p : X \to X/_\sim$, $p(x) = [x]$, is called the *quotient* or *identification map.*

Note. The identification map is onto and continuous. Indeed, the topology on $X/_\sim$ is the maximal (strongest/finest) topology for which p is continuous.

Example 5.8. Consider $\mathbb{R}$ and the equivalence relation $x \sim y$ if $x - y$ is an integer. Then $\mathbb{R}/_\sim$ is homeomorphic to S^1 via the homeomorphism $h : \mathbb{R}/_\sim \to S^1$ defined $h([x]) = e^{2\pi i x}$.

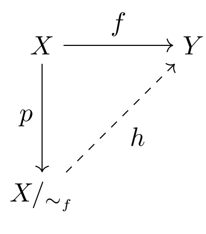
Consider E^2 and the equivalence relation $x \sim y$ if $x = y$ or $x, y \in S^1$. Then $E^2/_\sim$ is homeomorphic to the sphere S^2.

Consider $[0, 1] \times [0, 1]$ and the equivalence relation $(x, y) \sim (x', y')$ if $(x, y) = (x', y')$ or if $x = 0$, $x' = 1$, and $y = y'$. Then $[0, 1] \times [0, 1]/_\sim$ is homeomorphic to a cylinder.

Now, suppose that we have a continuous map $f : X \to Y$. We cam define an equivalence relation $\sim_f$ on X by $x \sim_f x'$ if $f(x) = f(x')$. Suppose that f is onto. A natural question is: "When is Y homeomorphic to $X/_{\sim_f}$?" Note that it is not always the case; for example, consider $X = [0, 1)$ and $Y = S^1$ and the map described earlier. However, the following result provides a positive answer.

Theorem 5.9. *If X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous onto map, then $X/_{\sim_f}$ is homeomorphic to Y.*

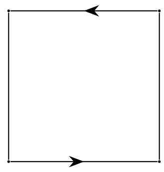
Proof. Let $h : X/_{\sim_f} \to Y$ be a function defined $h([x]) = f(x)$.



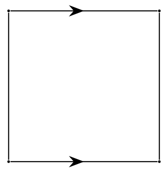
Then h is well-defined by definition, onto because f is onto, one-to-one because $h([x]) = h([y])$ implies $f(x) = f(y)$ such that $x \sim_f y$ and consequently $[x] = [y]$. Moreover, h is continuous because if O is open in Y then $f^{-1}(O)$ is open in X. But $f^{-1}(O) = (h \circ p)^{-1}(O)$ which equals $p^{-1}\big(h^{-1}(O)\big)$ is open in X, so $h^{-1}(O)$ is open by definition of the quotient topology. $X/_{\sim_f}$ is compact as it is a continuous image of a compact space. Thus, h is a homeomorphism. □

Note. From this theorem, it follows that $E^2/_\sim$ is homeomorphic to S^2, where $\sim$ is the equivalence relation that identifies all points in S^1 to a single point.

Example 5.10. A *Mobius strip* is the space obtained from $[0,1] \times [0,1]$ by identifying $(x,0)$ and $(1-x,1)$ for all $x \in [0,1]$.

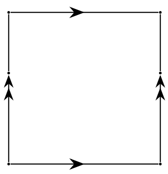


It can be shown that the Mobius strip is not homeomorphic to the cylinder. The cylinder has the following identification.

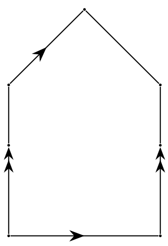


If you take a strip of paper and twist it twice, and glue the ends together, the resulting space is homeomorphic to a cylinder.

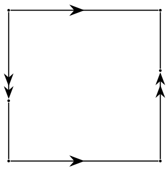
Example 5.11. The *torus* ($T = S^1 \times S^1$) is the space obtained from $[0,1] \times [0,1]$ by identifying $(x,0)$ with $(x,1)$ for all $x \in [0,1]$ and $(0,y)$ with $(1,y)$ for all $y \in [0,1]$.



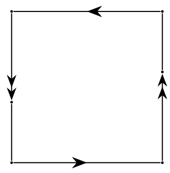
Consider the following, which is the torus with unit disc removed.



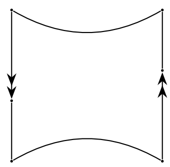
Example 5.12. The *Klein bottle* (K) is the space obtained from $[0,1] \times [0,1]$ by identifying $(x,0)$ with $(x,1)$ for all $x \in [0,1]$ and $(0,y)$ with $(1,1-y)$ for all $y \in [0,1]$.



Example 5.13. The *projective plane* P^2 is the topological space that corresponds to the (square) cell $[0,1] \times [0,1]$ under the identification map shown below.



P^2 with a disc removed is homeomorphic to the Mobius strip M. One can see this by the below.



Example 5.14. The Klein bottle K and the projective plane P^2 cannot be embedded in $\mathbb{R}^3$ (but they can in $\mathbb{R}^4$). Moreover, P^2 is the same topological space (up to homeomorphism) as the space obtained by any one of the following three procedures:

(1) Take the circular disc E^2, with the equivalence relation $x \sim x'$ if $x = x'$ or if x and x' lie on the boundary of E^2 and $x = -x'$ (antipodal point).
(2) Take the sphere S^2 with the equivalence relation $x \sim x'$ if $x = x'$ or $x = -x'$.
(3) Take $R^3 \setminus \{(0,0,0)\}$, with the equivalence relation $x \sim x'$ if x and x' lie on a line through the origin.

Note that the space described by (3) can be viewed as the 'space of lines through the origin', or more algebraically as the space of '1-dimensional subspaces of the 3-dimension vector space $\mathbb{R}^3$'. in this form, replacing $\mathbb{R}$ by $\mathbb{C}$, on similarly obtained a complex projective plane. Furthermore, $\mathbb{R}^3$ is a 3-dimensional vector space and the point set of the projective plane is a 1-dimensional vector subspace. We can define an equivalence relation on $\mathbb{R}^3 \setminus \{\mathbf{0}\} \sim$ by $x \sim y$ iff x, y are linearly dependent. With $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ equipped with the Euclidean topology, we obtain $\mathbb{R}^3 \setminus \{\mathbf{0}\}/_\sim \cong P^2$.

The (real) projective plane P^2 is compact and connected; $p : S^2 \to P^2$ onto continuous and S^2 is connected/compact, so P^2 is connected/compact.

Note. For $n \geq 2$, the space P^n (called *n-dimensional projective space* P^n) is defined exactly analogously, simply replacing ^{2} by ^{n}.

For P^3, there is a further way we can view this, namely as the space of rotations in $\mathbb{R}^3$; this space forms a group under composition (called $SO(3)$) and it is isomorphic to the group of 3×3 real matrices M having determinant 1 and with $MM^T = I$ under matrix multiplication. This 'topological group' plays an important role in physics.

To see how P^3 corresponds to the space of rotations in $\mathbb{R}^3$, recall that P^3 can be viewed as the three-dimensional ball E^3 under the equivalence relation that identifies antipodal points on the boundary sphere.

It can be shown that each rotation in $\mathbb{R}^3$ fixes some line through the origin. Thus if we take the radius of the ball E^3 to be π, each rotation in $\mathbb{R}^3$ can be viewed as a point p in this ball, where the line of rotation is the line passing through p and the origin, and the angle of clockwise rotation (facing the origin) is the distance of p from the origin.

There is another way to use functions to define a topological space - namely as a way to 'attach' Y to X by some function f defined on a subspace A of Y. This is just another type of quotient space.

Definition 5.15. Given two topological spaces X and Y, let $W = X \amalg Y$ denote the disjoint union of the spaces X and Y (note that this is a disconnected space). For a subspace A of Y, let $f : A \to X$ be a continuous function, and let

$$X \cup_f Y = X \amalg Y/_{\sim_f}$$

where $\sim_f$ is the equivalence relation on $X \amalg Y$ defined by setting

$$w \sim_f w' \iff \begin{cases} w = w'; & \text{or} \\ w, w' \in A \text{ and } f(w) = f(w'); & \text{or} \\ w \in A, w' \in X \text{ and } f(w) = w'; & \text{or} \\ w' \in A, w \in X \text{ and } f(w') = w. \end{cases}$$

The space $X \cup_f Y$ is called an *adjunction space.*

Note. Roughly speaking, the space $X \cup_f Y$ can be thought of as the space obtained by glueing Y onto X using the map f. Often we will take Y to be a cell E^n and $A = S^{n-1}$.

Example 5.16. Let G_n be the space obtained by taking $n \geq 1$ copies of S^1 and selecting a point in each of these spaces, and identifying these points to a single point (thus $G_1 = S^1$). The torus $T = S^1 \times S^1$ and the Klein bottle K can then both be written as adjunction spaces:

$$T = G_2 \cup_f E^2 \text{ and } K = G_2 \cup_g E^2,$$

where $f, g : S^1 \to G_2$ are suitable functions.

We can obtain the 2-sphere S^2 as an adjunction space of a point and a 2-cell: We have that $f : S^1 \to \{p\}$ identifying all points on S^1 to a single point p has $S^2 \cong \{p\} \cup_f E^2$.

We can write the projective plane P^2 as $P^2 = G_1 \cup_h E^2$ for function $h : S^1 \to G_1$ defined $h(z) = z^2$ (which is a 2-1 map), where $S^1 \subseteq \mathbb{C}$.

Definition 5.17. Many topological spaces can be built up by the following process by repeatedly forming adjunction spaces in which the boundaries of cells are glued to the (lower dimensional) spaces so far constructed (as in the examples above).

First, start with the discrete space X_0 consisting of disjoint points (i.e., 'zero-dimensional cells' $E^0 \amalg E^0 \ldots \amalg E^0$). Then attach (zero or more) 1-dimensional cells to X_0 to obtain a space X_1. Next, attach (zero or more) 2-dimensional cells to X_1 to obtain X_2, and so on. A space $X = X_n$ that can be constructed in this way is called a (finite dimensional) *CW complex*, where the 'C' stands for closure finite and the 'W' stands for weak topology (not in the 'coarse' sense).

Example 5.18. $\mathbb{R}$ can be described with $X_0 = \mathbb{Z}$ and the 1-dimensional cells of the form $[i, i+1]$ for each $i \in \mathbb{Z}$.

Note that if a finite number (say, n_i) of cells of dimension i are attached for $i = 0, 1, 2, \ldots$ then the resulting space is compact. Such a space is called a *finite CW complex*, or, more informally, a *finite cell complex.*

Associated with any topological space X is an important integer-valued invariant, called the *Euler characteristic* of X, and written $\chi(X)$. When X is a finite cell complex, then $\chi(X)$ has a simple description - it equals the number of cells of even dimension minus the number of cells of odd dimension in any cellular construction of X.

6. Manifolds, surfaces and knots

6.1. Manifolds.

Definition 6.1. For $n \geq 1$, an *n-manifold* X is a second countable Hausdorff space with the property that for each element $x \in X$, there is an open nhood O_x of x that is homeomorphic to the open Euclidean ball $\mathrm{Int}\,(E^n)$.

Note. An n-manifold is a topological space that is 'locally Euclidean'. For example, $\mathbb{R}$ and S^1 are both 1-manifolds, and it can be shown that S^1 is (up to homeomorphism) the unique compact connected 1-manifold, whereas the sphere S^2 and the torus $T = S^1 \times S^1$ are 2-manifolds.

We require the property of the space being second countable in order to avoid pathological examples such as the *long line* (i.e., the *Alexandroff line*).

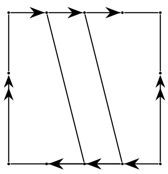
6.2. **Surfaces.** Compact connected 2-manifolds are often called *(closed) surfaces.* They include the sphere, the torus, the projective plane and the Klein bottle.

We will formally prove that these spaces are all topologically distinct (i.e., no two are homeomorphic)) later.

A central early question in topology was the following: "What other closed surfaces are possible?" Recall that a Mobius strip is the space obtained from $[0,1] \times [0,1]$ by identifying $(x, 0)$ and $(1-x, 1)$ for all $x \in [0,1]$. It helps to first classify closed surfaces into two types: Those that contain a subspace homeomorphic to the Mobius strip and those that do not. Surfaces of the first type are called *non-orientable*, the remainder are called *orientable.* P^2 and K are examples of non-orientable closed surfaces, while S^2 and T are examples of orientable surfaces. Note that every closed surface is (by definition) either orientable or not (and cannot be both).

Example 6.2. The Klein bottle K not only contains an embedding of the Mobius strip, K can be obtained by glueing two copies of a Mobius strip together along their boundary (each of which is homeomorphic to S^1).

Proof. Consider the identification map on $[0,1] \times [0,1]$ for K:



which can be cut along the two diagonal lines into three pieces, and then put back together to obtain two copies of a Mobius strip. □

If M and N are two closed surfaces, remove an open disc from M and an open disc from N and then identify the two resulting boundaries (which are homeomorphic to S^1) under any homeomorphism from one boundary to the other, to obtain a new closed surface. It turns out that the operation # is well-defined, in the sense that the resulting closed surface does not depend (up to homeomorphism) on the choice of discs removed, or the homeomorphism used to identify the boundary. Connected sum is an example of what in topology is called *surgery.* If we take $N = M \# T$, where T is the torus, we say that N is obtained from M by *adding a handle.*

It can be shown that

- # is associative
- # is commutative
- # has an identity element (S^2)

Thus, the surfaces form a commutative semigroup under #, with identity (S^2). If N is non-orientable and O is orientable, then $N\#O$ is non-orientable.

Theorem 6.3. (1) *Any orientable closed surface is homeomorphic either to the sphere or to the sphere with $n \geq 1$ handles added.*
(2) *Any non-orientable closed surface is homeomorphic to the projective plane P^2, or the Klein bottle K or to P^2 with $n \geq 1$ handles added.*
(3) *No two of these surfaces are homeomorphic.*

In fact, we can describe the semigroup structure of closed surfaces very explicitly. If we let P be the projective plane (denoted P^2 earlier) then $P\#P \cong K$ and $P\#P\#P \cong P\#T$, which can also be written as $P\#K \cong P\#T$. Thus another (equivalent) way to classify closed surfaces is as follows: For $g \geq 0$ let, T_g be the space obtained by adding g handles to S^2 (thus $T_0 \cong S^2$, $T_1 \cong T$, and $T_g \cong T\#T \dots \#T$ (g times)). Then every closed surface is either T_g for some $g \geq 0$ (if it is orientable) or it is a connected sum of $k \geq 1$ projective planes (if it is non-orientable).

Proposition 6.4. *For the Euler characteristic of closed surfaces M and N:*

$$\chi(M\#N) = \chi(M) + \chi(N) - 2.$$

Proof. Consider a cell complex description of the surface with boundary M' obtained from M by deleting open 2-cell. Then M can be obtained from M' by adding a 2-cell (E^2) to $\partial M'$. So, $\chi(M) = \chi(M') + 1$. Similarly, $\chi(N) = \chi(N') + 1$. Now consider a cell description of $M\#N$ starting with one 0-cell and attaching a 1-cell to form S^1, and attach cells to this to form M' and cells to this to form N'. Counting the total number of cells of dimensions $0, 1, 2$ we get $\chi(M\#N) = \chi(M') + \chi(N')$. □

Theorem 6.5. *A compact connected 2-manifold M is characterised up to homeomorphism by specifying its Euler characteristic and stating whether or not it is orientable.*

Proof. If M is orientable, then $M = T_g$ for some $g \geq 0$. Since $\chi(T) = 0$ and $\chi(S^2) = 2$, it follows $\chi(T_g) = 2 - 2g$, and so $\chi(T_g)$ determines g, and thus the surface up to homeomorpism.

If M is non-orientable, then M is a connected sum of $k \geq 1$ projective planes and so $\chi(M) = 2 - k$ (since $\chi(P^2) = 1$), and so $\chi(M)$ determines k and therefore M. □

Notice that any orientable closed surface can be embedded in $\mathbb{R}^3$. It can be shown that no non-orientable closed surface can be embedded in $\mathbb{R}^3$ (but it can in $\mathbb{R}^4$).

Definition 6.6. A n-dimensional *manifold-with-boundary* is a Hausdorff topological space X with the property that for each element $x \in X$, there is an open set O_x containing x that is homeomorphic to the half-open Euclidean ball $\text{Int}(E^n_+) = \{(x_1, \dots, x_n) \in \text{Int}(E^n) \mid x_n \geq 0\}$. Let X° be the points in X having an open set homeomorphic to $\text{Int}(E^n)$. Then $\partial X = X \setminus X^\circ$ is called the *boundary* of X, and is a topological invariant.

Example 6.7. The Mobius strip and cylinder are not homeomorphic because they do not have the same boundary. The cylinder has boundary $S^1 \coprod S^1$ whereas the Mobius strip has boundary S^1. The Mobius strip cannot be embedded in $\mathbb{R}^2$.

6.3. **Compact and connected manifolds of higher dimension.** Examples of compact and connected 3-manifolds include S^3, P^3 and $S^1 \times M$, where M is any closed surface. However, there are many other examples, and there study and classification is much more complex than for closed surfaces (indeed the famous Poincaré conjecture involves this topic).

Theorem 6.8. *Suppose M and N are connected n-dimensional manifolds, and that M is compact. Then M cannot be embedded in N unless N is homeomorphic to M (and in that case show that any embedding is onto).*

Note. The proof requires the following famous result in topology called the *Invariance of Domain Theorem*:

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one continuous function. Then f is open.

Proof. Suppose $f : M \to N$ is an embedding. Then $f(M)$ is compact in the Hausdorff space N, and so it follows $f(M)$ is closed in N. We now show that $f(M)$ is also open in N, and since N is connected this implies that $f(M) = N$ (in which case M and N are homeomorphic).

To show that $f(M)$ is open in N, each point x in M has an open nhood $O_x \cong \mathbb{R}^n$, and $f(x)$ has an open nhood $O'_{f(x)} \cong \mathbb{R}^n$. Moreover, we may assume that $f(O_x) \subseteq O'_{f(x)}$. Then $f(O_x)$ is open in $O'_{f(x)}$ by the invariance of domain theorem, and so $f(O_x)$ is open in N, and so every point in $f(M)$ is contained in an open set in N. So $f(M)$ is open in N. □

6.4. **Knots.** A *knot* is the image $k = f\big(S^1\big)$ of an embedding f of S^1 into $\mathbb{R}^3$.

The circle S^1, as it is usually represented, is one example of a knot (called the 'unknot'). Two knots k_1 and k_2 are regarded as *equivalent* if there is an *ambient isotopy* from k_1 to k_2 (written $k_1 \sim k_2$). This phrase refers to a continuous map

$$F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

for which $F(x,0)$ is the identity map, $F(k_1,1) = k_2$, and where, for each $0 \le t \le 1$, $F(x,t)$ is a homeomorphism from $\mathbb{R}^3$ to itself. Note that $F(k_1) = k_2$ means $\{F(x,1) \mid x \in k_1\} = k_2$.

Thus F continuously moves $\mathbb{R}^3$ around so as to transform knot k_1 (at time $t = 0$) into k_2 (at time $t = 1$).

Proposition 6.9. *The relation $\sim$ is an equivalence relation on the set of knots.*

Proof. For $F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ we write $F(x,t) = F_t(x)$, where $F_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism for each t. $F_0 = 1_{\mathbb{R}^3}$ and F_1 maps the set k_1 to k_2. Let G be the required ambient isotopy to establish reflexivity, symmetry and transitivity.

To show reflexivity, take $G(x,t) = x$ for each $x \in \mathbb{R}^3$ and $t \in [0,1]$.

To show transitivity, suppose $k_1 \sim^F k_2 \sim^{F'} k_3$. One can take

$$G(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ F'(F(x,1),2t-1) & \text{if } \frac{1}{2} \le t \le 1; \end{cases}$$

or alternatively $G(x,t) = F'(F(x,t),t)$ for each $t \in [0,1]$.

To show symmetry, suppose $k_1 \sim^F k_2$ (by F). Let $G : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ be $G_t(x) = F_t^{-1}(x)$. Then $G_0(x) = x$ for each $x \in \mathbb{R}^3$, that is, $G_0 = 1_{\mathbb{R}^3}$, and $G_1(k_2) = k_1$. We need to show that G is continuous. Define $H : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$H(x,t) = \begin{cases} (x,t) & \text{if } (x,t) \in \mathbb{R}^3 \times (-\infty,0]; \\ (F_t(x),t) & \text{if } (x,t) \in \mathbb{R}^3 \times [0,1]; \\ (F_1(x),t) & \text{if } (x,t) \in \mathbb{R}^3 \times [1,\infty). \end{cases}$$

H is continuous by the glueing lemma and is one-to-one, and so, by the invariance of domain theorem, H is open. Thus the restriction H' of H to $\mathbb{R}^3 \times [0,1]$ ($H'(x,t) = (F_t(x), t)$) is continuous, open, $1-1$, and onto, so it is a homeomorphism. Thus its inverse $\hat{G}(x,t) = \big(F_t^{-1}(x), t\big)$ is continuous, and so the function $G = \pi_1 \circ \hat{G}$ is continuous (where π_1 is the projection map). □

Note. It might seem that a simpler definition of when two knots k_1, k_2 are equivalent is the following. Let's say k_1 and k_2 are 'similar' if there exists a continuous function $F : S^1 \times [0,1] \to \mathbb{R}^3$ so that $F\big(S^1, 0\big) = k_1$ and $F\big(S^1, 1\big) = k_2$ and $F(x,t)$ is an embedding for all $t \in [0,1]$. However, this definition turns out not to be a good one (essentially because all (tame) knots are equivalent, for example, the trefoil knot and unknot).

Definition 6.10. A knot k that is equivalent to the usual ('un-knotted') embedding of S^1 in $\mathbb{R}^3$ is often called the *unknot.*

A knot that is not equivalent to its mirror image is called a *chiral knot.* The trefoil knot is an example of a chiral knot.

Proposition 6.11. *If k_1 and k_2 are equivalent knots, then $\mathbb{R}^3 \setminus k_1$ is homeomorphic to $\mathbb{R}^3 \setminus k_2$.*

Proof. Suppose $k_1 \sim^F k_2$. Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be $h(x) = F(x,1)$. Then h is a homeomorphism and since h maps k_1 bijectively to k_2, the restriction h' of h to $\mathbb{R}^3 \setminus k_1$ is a homeomorphism from $\mathbb{R}^3 \setminus k_1$ to $\mathbb{R}^3 \setminus k_2$. □

Note. The converse is not necessarily true (e.g., if the knot is chiral). However it is *almost* true: a (hard) theorem of Gordon and Luecke is that for any two knots k_1 and k_2, if $\mathbb{R}^3 \setminus k_1$ is homeomorphic to $\mathbb{R}^3 \setminus k_2$, then either k_2 is equivalent to k_1, or else k_2 is the mirror image of k_1.

Thus the classification of knots up to equivalence comes down to classifying the topology of the 3-dimensional manifolds $\mathbb{R}^3 \setminus k$.

7. Homotopy and retractions

Definition 7.1. Given a topological space X and a subspace $A \subseteq X$, a continuous map $r : X \to A$ is said to be a *retraction* (from X to A) if $r(x) = x$ for all $x \in A$. When such a map exists, we say that A is a *retract* of X. Retractions are idempotent with respect to function composition, i.e., $r \circ r = r$.

Another way of describing this is to say that $r \circ i$ is the identity function on A where $i : A \to X$ is the inclusion map that sends $x \in A$ to the same point $x \in X$.

This can be represented by

$$\underbrace{A \overset{i}{\hookrightarrow} X \overset{r}{\to} A}_{r \circ i = 1_A}.$$

Example 7.2. S^1 is a retract of $E^2 \setminus \{p\}$ for any point p in the interior of E^2. We prove the more general case, which is noted below.

Proof. Draw a line p to $x \in E^n \setminus \{p\}$ and extend this line until it hits the boundary of E^n at some point x^*. Then let $r(x) = x^*$. This function r is continuous and fixes each point of $S^{n-1} = \partial E^n$. □

More generally, for $n \geq 1$, there is a retraction from $E^n \setminus \{p\}$ to its boundary sphere S^{n-1} for any point p in the interior of E^n.

Example 7.3. There is a retraction from the cylinder $C = S^1 \times [0,1]$ onto half of its boundary, but not to its full boundary.

Proof. Not on full boundary since C is connected, yet its boundary is not. □

The following result can be proved easily in dimension $n = 1$ and we will establish it later in the case $n = 2$, but its general proof requires more advanced techniques.

Theorem 7.4. *There is no retraction from E^n to S^{n-1} for any $n \geq 1$.*

Corollary 7.5. (The Brouwer fixed point theorem ($n \leq 2$)). *Any continuous map f from E^n to itself has a fixed point.*

Proof. The case $n = 1$ follows from the intermediate value theorem, so suppose $n \geq 2$. Suppose that $f : E^n \to E^n$ has no fixed point. Consider the function $r : E^n \to S^{n-1}$, defined as follows. Draw a ray from $f(x)$ to x and continue until it hits a point $x^* \in S^{n-1}$, where we let $r(x) = x^*$. Then r is continuous and r fixes points on S^{n-1}. So r is a retraction from E^n to S^{n-1}. But this contradicts the theorem above. □

Theorem 7.6. *Let A be a $n \times n$ matrix with non-negative entries. Then A has non-negative eigenvalue.*

Proof. Suppose that A is singular. Then $Ax = 0$ has a non-zero solution x, so $\lambda = 0$ is a non-negative eigenvalue for A. Otherwise, $Ax \neq 0$ for all $x \neq 0$. Consider S^{n-1} as the unit sphere centred on 0, and let B be the portion of the sphere in the positive quadrant. The set $B \cong E^{n-1}$ and let $f : E^{n-1} \to E^{n-1}$ be defined by $f(x) = \frac{Ax}{\|Ax\|}$. Then

- f is well-defined, since $Ax \neq 0$ on B, and
- f is continuous.

By the Brouwer fixed point theorem, there exists $x^* \in B$ such that $f(x^*) = x^*$. Hence,

$$\begin{aligned} & \frac{Ax^*}{\|Ax^*\|} = x^* \\ \implies \quad & Ax^* = \underbrace{\|Ax^*\|}_{\lambda > 0} \underbrace{x^*}_{>0}. \end{aligned}$$

□

Proposition 7.7. *If any continuous map f from E^n to itself has a fixed point, then there is no retraction from E^n to S^{n-1} (for any $n \geq 1$).*

Proof. Suppose $r : E^n \to S^{n-1}$ is a retraction. Let $g : S^{n-1} \to E^n$ be the antipodal map $g(x) = -x$. Then $g \circ r : E^n \to E^n$ is continuous, and has no fixed points. □

7.1. **Homotopy.**

Definition 7.8. Two maps $f, g : X \to Y$ are *homotopic* to each other if there is a continuous function

$$F : X \times [0, 1] \to Y,$$

for which $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for each $x \in X$. Informally, f and g are homotopic if f can be continuously deformed into the function g. We will write $f \sim g$ to denote that f and g are homotopic and we will often write the name of the homotopy function (F) over the top of the $\sim$ symbol.

Example 7.9.

1. Let $X = Y = \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$, $g(x^3)$. Then $f \sim g$ via $F(x, t) = x^{2+t}$.
2. Let $X = Y = \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be any two continuous functions. Then $f \sim g$ via $F(x, t) = tg(x) + (1 - t)f(x)$.
3. Let $X = Y = S^1$, $f = 1_X$ and g be the antipodal map. We consider 'rotating' by 180°. Regard S^1 as the unit circle in $\mathbb{C}$ centred by 0 and let $F(x, t) = xe^{i\pi t}$.

(4) Using the same set up as (3) except on S^2, we can prove there is no way (it turns out $f \not\sim g$). But it is for S^3.

Proposition 7.10. *The relation defined by $f \sim g$ if f and g are homotopic is an equivalence relation.*

Proof. (1) $f \sim f$. Take $F(x,t) = f(x)$ for all $t \in [0,1]$.

(2) $f \overset{F}{\sim} g$ then $g \overset{F'}{\sim} f$. Take $F'(x,t) = F(x, 1-t)$.

(3) $f \overset{F}{\sim} g$ and $g \overset{G}{\sim} h$, then $f \overset{H}{\sim} g$, where

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2}. \\ G(x,2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

By the glueing lemma, H is indeed continuous.

□

Definition 7.11. Two spaces X and Y are *homotopy equivalent* (or *homotopic* to each other) if there are maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f : X \to X$ is homotopic to the identity map $1_X : X \to X$ and $f\circ : Y \to Y$ is homotopic to the identity map $1_Y : Y \to Y$. When this holds, we write $X \approx Y$.

Note. If X and Y are homeomorphic, then X and Y are homotopy equivalent since in this case $f \circ g$ and $g \circ f$ are equal to the identity maps on X and Y, respectively. Thus homotopic equivalence is a looser notion than saying the two spaces are homeomorphic.

Example 7.12. The cylinder C and the Mobius strip M are homotopy equivalent (but not homeomorphic). We can consider $f : S^1 \to C$ and $g : C \to S^1$ where f is the inclusion map and g is the retraction (by projecting). Then $g \circ f \sim 1_{S^1}$ and $f \circ g \sim 1_C$, so $C \approx S^1$. We also have that $M \approx S^1$, so by the result below, indeed we get $C \approx M$.

Proposition 7.13. *The relation defined by $X \approx Y$ if X and Y are homotopic is an equivalence relation.*

Proof. Both reflexivity and symmetry are trivial. To show transitivity, suppose

$$X \underset{f'}{\overset{f}{\rightleftarrows}} Y \underset{g'}{\overset{g}{\rightleftarrows}} Z.$$

Then

$$X \underset{f' \circ g'}{\overset{g \circ f}{\rightleftarrows}} Z,$$

where

$$\begin{aligned} (f' \circ g') \circ (g \circ f) &\sim f' \circ ((g' \circ g) \circ f) \\ &\sim f' \circ (1_Y \circ f) \\ &\sim f' \circ f \\ &\sim 1_X, \end{aligned}$$

and $(g \circ f) \circ (f' \circ g') \sim 1_Z$ similarly. □

It can be shown that the Euler characteristic is a homotopy invariant for finite cell complexes.

Proposition 7.14. $\mathbb{R}^n$ *and* $\mathbb{R}^{n+1}$ *cannot be homeomorphic.*

Proof. Suppose $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ were homeomorphic. Then $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^{n+1} \setminus \{0\}$ are homeomorphic. Now, $\mathbb{R}^n \setminus \{0\}$ is homotopic to S^{n-1}. To see why, let $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ be $x \mapsto \frac{x}{\|x\|}$. Then r is a retraction, so it follows $r \circ i_{S^{n-1}} = 1_{S^{n-1}}$ and $1_{S^{n-1}}$ is homotopic to $1_{\mathbb{R}^n}$ by the homotopy $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ given by $(x,t) \mapsto tx + (1-t)r(x)$. But then $S^n \approx S^{n-1}$ which implies $\chi(S^n) = \chi\big(S^{n-1}\big)$ (which is a contradiction). □

Lemma 7.15. *Suppose that $f, f' : X \to Y$ and $g, g' : Y \to Z$ are continuous function that satisfy $f \sim f'$ and $g \sim g'$. Then $(g \circ f) \sim (g' \circ f')$.*

Proof. Suppose $f \overset{F}{\sim} f'$ and $g \overset{G}{\sim} g'$. Then $(g \circ f) \overset{H}{\sim} (g' \circ f')$ for $H(x,t) = G(F(x,t),t)$ (which is continuous) and $H(x,0) = (g \circ f)(x)$ and $H(x,1) = (g' \circ f')(x)$ for all $x \in X$. □

7.2. **Contractable spaces and maps.**

Definition 7.16. A space X is *contractable* if it is homotopy equivalent to the topological space consisting of a single point. Informally. contractable means that we can continuously 'squash down' X to a point.

Example 7.17. We now provide some examples of contractable spaces. Trivially, a single point is contractable. So are the following:

- Both E^n and $\operatorname{Int}(E^n)$ are contractable,
- $\mathbb{R}^n$ is contractable,
- 1-dimensional cell complexes are contractable iff it is a tree (otherwise can get homotopy to G_k).

Note that S^n is not contractable for all $n \geq 1$.

Proposition 7.18. *Any contractable space is (path) connected.*

Proof. Let $x, y \in X$. Let $F : X \times [0,1] \to X$ be a homotopy from 1_X to C_{x_0}. Then $F(x,t)$ is a path from x to x_0, and $F(y,t)$ is a path from y to x_0. So, x and y are connected by the first path followed by the second path in reverse. That is,

$$p(t) = \begin{cases} F(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ F(y, 2-2t) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

□

The *cone* of X is the quotient space $CX = X \times [0,1]/_\sim$, where $(x,t) \sim (x',t')$ iff $(x,t) = (x',t')$ or $t = t' = 1$.

Proposition 7.19. *For any topological space X, its cone is contractable.*

Proof. Define $F : CX \times [0,1] \to CX$ as $F([(x,s)],t) = [(x,(1-t)s+t)]$. Then $1_{CX} \overset{F}{\sim} c_{x_0}$. □

Proposition 7.20. *A topological space X is contractable if and only if X is a retract of CX, where we identify X with the subspace $X \times \{0\}/_\sim$ of CX.*

Proof. Suppose X is contractable. Then for $x_0 \in X$ there is a homotopy F from 1_X to c_{x_0} ($x \mapsto x_0$ for all $x \in X$). Let $r_F([(x,s)]) = [(F(x,s),0)]$, so

- $r_F([(x,0)]) = [(x,0)]$,
- $r_F([(x,1)]) = [(x_0,1)]$,

for all $x \in X$.

Conversely, suppose X is a retract of CX. Then CX is contractable. We will show more generally that if Y is contractable and X is a retract of Y (via r) then X is contractable too. Given a homotopy F from 1_Y to the constant map c_{y_0} on

Y, define $G : X \times [0,1] \to X$ by $G(x,s) = r(F(x,s))$. Then G is a homotopy from 1_X to the constant map for X $(x \mapsto r(y_0))$. □

Proposition 7.21. *A subset A of $\mathbb{R}^n$ is said to be* convex *if the line joining any two points in A lies within A. Any convex subset of $\mathbb{R}^n$ is contractable.*

Proof. Follows from second assignment. □

7.3. **Deformation retracts.** A particular type of homotopy equivalence arises when a subspace A of X is a retract of X (i.e., there is a continuous map $r : X \to A$ with $r(a) = a$ for all $a \in A$), and the retraction r is homotopic to the identity map on X. When this holds we say that A is a *deformation retract* of X.

Thus X is contractable if and only if $A = \{x_0\}$ is a deformation retract of X for some $x_0 \in X$ (i.e., the identity map on X is homotopic to a constant map $r(x) = x_0$ for all $x \in X$).

Proposition 7.22. *If A is a deformation retract of X, then $X \approx A$.*

Proof. Observe that

$$\underbrace{A \overset{i}{\hookrightarrow} X \overset{r}{\to} A}_{r \circ i = 1_A}$$

and

$$\underbrace{X \overset{r}{\to} A \overset{i}{\hookrightarrow} X}_{i \circ r \sim 1_X},$$

which implies X and A are homotopic. □

Equivalently, the subspace A is a deformation retract of X precisely if there is a continuous map

$$F : X \times [0,1] \to X$$

with

- $F(x,0) = x$ for all $x \in X$,
- $F(x,1) \in A$ for all $x \in X$, and
- $F(a,1) = a$ for all $a \in A$.

If, in addition $F(a,s) = a$ for all $a \in A$ and $s \in [0,1]$, we say that A is a *strong deformation retract* of X. Informally, this stronger notion means the space X can be continuously squashed down onto A without moving any points in A during this squashing.

Example 7.23. We provide an example of a retraction that is not a deformation retract. Consider $r : S^1 \times [0,1] \to \{p\} \times [0,1]$ defined $r(x,s) = (p,s)$ (for some fixed $p \in S^1$). If r was a deformation retract, then $\{p\} \times [0,1] \approx S^1 \times [0,1]$. But then

$$1 = \chi(\{p\} \times [0,1]) = \chi\big(S^1 \times [0,1]\big) = 0,$$

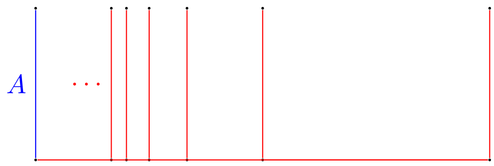
a contradiction.

Proposition 7.24. *If X is a strong deformation retract of Y and W is a strong deformation retraction of Y, then $X \approx W$.*

Proof. Follows from transitivity ($X \approx Y$ and $W \approx Y$ implies $X \approx W$). □

Example 7.25. We prove the comb space is an example of a space which satisfies the property: a subspace A that is a deformation retract of another space X but not a strong deformation retract of X.

Proof. Consider the comb space $X = [0,1]\times\{0\}\cup\{0\}\times[0,1]\cup\left(\bigcup_{n\geq 1}\left\{\frac{1}{n}\right\}\times[0,1]\right)$ and $A = \{0\}\times[0,1]$, shown below.



Comb space

Let $r : X \to A$ be defined $r(x,y) = (0,y)$ for all $x, y \in X$. Then r is a retraction from x to A, and $i \circ r \sim 1_X$ since

(1) $1_X \overset{F_1}{\sim} c$, where $c(x) = (0,0)$ for all $x \in X$ by

$$F_1((x,y),t) = \begin{cases} (x,(1-2t)y) & 0 \leq t \leq \frac{1}{2}; \\ ((2-2t)x, 0) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

(2) $i \circ r \overset{F_2}{\sim} c$, where $F_2((x,y),t) = (0,(1-t)y)$.

By transitivity, $i \circ r \sim 1_X$. Thus, A is a deformation retract of X.

Now, suppose A was a strong deformation retract of X via some F. Let $x_n = \left(\frac{1}{n},1\right)$, $x'_n = \left(\frac{1}{n},0\right)$ and $x = (0,1)$. Then $p_n(t) = F(x_n,t)$ where $t \in [0,1]$ is a path from x_n to x, and passes through x'_n. Let t_n be the first value of $t \geq 0$ for which $F(x_n,t) = x'_n$ (t_n exists by compactness argument, since $p_n^{-1}(x'_n)$ is a non-empty closed subset of $[0,1]$ and so is a compact subset of $[0,1]$). Now, $d(x_n,x) \to 0$ as $n \to \infty$ and so $d((x_n,t_n),(x,t_n)) \to 0$ as $n \to \infty$. F is uniformly continuous on $X \times [0,1]$ since this space is closed, bounded subspace of $\mathbb{R}^3$ and so compact. Thus $d(F(x_n,t_n),F(x,t_n)) \to 0$ as $n \to \infty$. But $F(x_n,t_n) = x'_n$ and $F(x,t_n) = x = (0,1)$, where $x'_n \to (0,0)$ as $n \to \infty$ (a contradiction). □

8. Introduction to algebraic topology

Let X be a topological space and let x_0 be a fixed point in X. A continuous function $\alpha : [0,1] \to X$ satisfying $\alpha(0) = \alpha(1) = x_0$ is called a *loop based at* x_0. Given two loops based at x_0, say α and β, we can form a new loop at x_0, denoted $\alpha \cdot \beta$, by first following α (at twice the speed) then following β at twice the speed. More precisely:

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2}; \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Define a relation $\sim$ on loops based at x as follows. We say that γ is *homotopic to* γ' *relative* to $0,1$ (denoted $\gamma \sim \gamma'$ rel $(0,1)$) if there is a homotopy $F : I \times I \to X$ with: $F(s,0) = \gamma(s)$, $F(s,1) = \gamma'(s)$ and $F(0,t) = F(1,t) = x_0$ for all $t \in [0,1]$. In other words, we can continuously deform the loop γ into the loop γ' while keeping the start and endpoints of the loop fixed at x_0.

It can be checked that the relation $\sim$ rel $(0,1)$ is an equivalence relation, so we will let $[\gamma]$ denote the equivalence class of γ. The operation $\cdot$ now induced a well-defined and associative binary operation on the set of equivalence classes of loops based at x_0, by defining $[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$. Moreover, there is an identity element $[\iota]$ for this binary operation (take the constant map to x_0), and each element $[\gamma]$ has an inverse. Thus, the equivalence classes of loops based at x_0 forms a group under this binary operation. This group is denoted $\pi_1(X,x_0)$ and is called the *fundamental group* of (X,x_0).

Proposition 8.1. *If X is path-connected, and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic groups.*

Proof. Let α be a path from x_0 to x_1, and let $\overline{\alpha}$ be the reverse of this path (i.e., $\overline{\alpha}(s) = \alpha(1-s)$ for all $s \in [0,1]$).

Let $h_\alpha : \pi_1(X, x_0) \to \pi_1(X, x_1)$ be defined by $h_\alpha([\gamma]) = [\overline{\alpha} \cdot \gamma \cdot \alpha]$. Then h_α is well-defined and a group homomorphism. Since $h_{\overline{\alpha}}$ is an inverse for h_α, it follows that h_α is an isomorphism. □

Note. For a path-connected space, we will sometimes just denote the group $\pi_1(X, x_0)$ as $\pi_1(X)$. A topological space is *simply connected* if it is path-connected and $\pi_1(X)$ is the trivial group.

Proposition 8.2. *A path-connected topological space X is simply connected if and only if, for all $x \neq y \in X$ and any two paths γ and γ' from x to y, there is a homotopy from γ to γ' that 'fixes x at 0 and y at 1'.*

Proof. ($\Longrightarrow$) Suppose X is simply connected and $x \neq y \in X$. Then $\overline{\gamma'} \cdot \gamma' \sim \iota_y$ rel $(0,1)$, so

$$\gamma \sim \gamma \cdot \left(\overline{\gamma'} \cdot \gamma'\right) \sim \left(\gamma \cdot \overline{\gamma'}\right) \cdot \gamma' \sim \iota_x \cdot \gamma' \sim \gamma',$$

where $\gamma \cdot \overline{\gamma'} \sim \iota_x$ because X is simply connected.

($\Longleftarrow$) Given a loop γ based at x, let $y = \gamma(0.5)$, and let γ_1, γ_2 be the two paths from x to y and y to x. So $\gamma = \gamma_1 \cdot \gamma_2$ and $\gamma_1 \sim \overline{\gamma_2}$ rel $(0,1)$ and so $[\gamma] = [\iota_x]$. □

Proposition 8.3. *A contractable space is simply connected.*

Proof. Suppose X is a contractable space. Let F be the homotopy between 1_X and x_0. If $\gamma : I \to X$ is any loop based at x_0, then $H : I \times I \to X$ defined $H(s,t) = F(\gamma(s), t)$ gives us that $\gamma \sim \iota_{x_0}$ relative $0, 1$. Hence, $\pi_1(X, x_0)$ is trivial. To see why X is path-connected, one needs only observe that we have a path from any point $x \in X$ to x_0. □

Note. Not every simply connected space is contractable, for example, take S^2.

Corollary 8.4. *$\pi_1(\mathbb{R}^n)$ is the trivial group. Similarly for E^n (the closed ball of dimension n).*

8.1. **Maps between spaces induce group homomorphisms.** Now, suppose we have a continuous function $g : (X, x_0) \to (Y, y_0)$; this means that $g : X \to Y$ is continuous and $g(x_0) = y_0$. Then g induces a group homomorphism g_* from the group $\pi_1(X, x_0)$ to the group $\pi_1(Y, y_0)$ defined by

$$g_*([\gamma]) = [g(\gamma)].$$

Proposition 8.5. *g_* is well-defined and is a homomorphism.*

Proof. If $\gamma \overset{F}{\sim} \gamma'$ rel $(0,1)$ then $g(\gamma) \overset{G}{\sim} g(\gamma')$ rel $(0,1)$ where $G = g \circ F$ (so g_* is well-defined).

Observe that

$$\begin{aligned} g_*([\gamma_1] \cdot [\gamma_2]) &= g_*([\gamma_1 \cdot \gamma_2]) \\ &= [g(\gamma_1 \cdot \gamma_2)] \\ &= [g(\gamma_1) \cdot g(\gamma_2)] \\ &= [g(\gamma_1)] \cdot [g(\gamma_2)] \\ &= g_*([\gamma_1]) \cdot g_*([\gamma_2]), \end{aligned}$$

which holds for all $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$. Thus, g_* is indeed a homomorphism. □

Lemma 8.6. (1) *If 1_X is the identity function on X, then $(1_X)_*$ is the identity function on $\pi_1(X, x_0)$.*

(2) *If $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (W, w_0)$ are continuous, then $(g \circ f)_* = g_* \circ f_*$.*

(3) *If $f, g : X \to Y$ are continuous and homotopic, and X, Y are path-connected, then for any $x_0 \in X$, we have $h_* \circ f_* = g_*$ for an isomorphism*

$$h_* : \pi_1(Y, f(x_0)) \to \pi_1(Y, g(x_0)).$$

Proof. Observe that

$$\begin{aligned}(g \circ f)_*[\gamma] &= [(g \circ f)(\gamma)] \\ &= [g(f(\gamma))] \\ &= g_*[f(\gamma)] \\ &= g_*(f_*[\gamma]) \\ &= (g_* \circ f_*)[\gamma],\end{aligned}$$

proving (2).

Suppose that $f \overset{F}{\sim} g$ $(F : X \times [0, 1] \to Y)$. Let $\alpha(t) = F(x_0, t)$ be the resulting path in Y from $f(x_0)$ to $g(x_0)$, and let $h_* = h_\alpha$ be the resulting isomorphism from $\pi_1(Y, f(x_0))$ to $\pi_1(Y, g(x_0))$, as defined in Proposition 8.1. Then for any loop γ based at x_0, $g(\gamma)$ and $\overline{\alpha} \cdot f(\gamma) \cdot \alpha$ are loops based at $g(x_0)$. We claim that $g(\gamma)$ and $\overline{\alpha} \cdot f(\gamma) \cdot \alpha$ are homotopic rel $(0, 1)$. For using this result, we get that

$$h_*([f(\gamma)]) = [\overline{\alpha} \cdot f(\gamma) \cdot \alpha] = [g(\gamma)]$$

for all γ, so $h_* \circ f_* = g_*$ (as desired).

Now, to prove the claim: Let

- $\beta_t(s) : [0, 1] \to Y$ be the path from $\alpha(1 - t)$ to $g(x_0)$ following path α,
- $\overline{\beta}_t(s) = \beta_t(1 - s)$, and
- $G(s, t) = \overline{\beta}_t(s) \cdot F(\gamma(s), 1 - t) \cdot \beta_t(s)$.

Then G is a homotopy from $g(\alpha)$ to $\overline{\alpha} \cdot f(\gamma) \cdot \alpha$ since

$$G(s, 0) = \overline{\beta}_0(s) \cdot F(\gamma(s), 1) \cdot \beta_0(s) \sim g(\gamma(s))$$

and

$$G(s, 1) = \overline{\beta}_1(s) \cdot F(\gamma(s), 0) \cdot \beta_1(s) \sim \overline{\alpha}(s) \cdot f(\alpha(s)) \cdot \alpha(s) \text{ rel } (0, 1),$$

proving (3). □

Corollary 8.7. *If $(X, x_0) \cong (Y, y_0)$ are homeomorphic then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ are isomorphic.*

Proof. There exists maps $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (X, x_0)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. This implies that $g_* \circ f_* = 1_{\pi_1(X, x_0)}$ and $f_* \circ g_* = 1_{\pi_1(Y, y_0)}$, giving us that f_* is an isomorphism. □

Corollary 8.8. *If path-connected spaces X and Y are homotopy equivalent spaces, then $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic groups. In particular, if X and Y are homeomorphic spaces then $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic groups.*

Note. (1) Two (path-connected) topological spaces with non-isomorphic groups cannot be homeomorphic to each other.

(2) If A is a deformation retract of a path-connected space X, then A and X have isomorphic fundamental groups.

(3) If $X \approx Y$ and X is simply connected, then so is Y.

Proof. If $X \approx Y$ then there exists maps $g : X \to Y$, $g' : Y \to X$ with $g' \circ g \sim 1_X$ and $g \circ g' \sim 1_Y$. Let

- $y_0 \in Y$,
- $x_0 = g'(y_0)$,
- $y_1 = g(x_0)$,
- $x_1 = g'(y_1)$.

Since $g' \circ g \sim 1_X$, $(g' \circ g)_*$ is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$. Now, $(g' \circ g)_* = g'_* \circ g_*$, so g_* is one-to-one. By similar argument, $(g \circ g')_*$ is an isomorphism from $\pi_1(Y, y_0)$ to $\pi_1(Y, y_1)$ and so $(g \circ g')_* = g_* \circ g'_*$ implies g_* is onto. Thus, g_* is an isomorphism. □

Proposition 8.9. *Suppose f, g are homotopic maps from X to Y with $f(x_0) = g(x_0) = y_0$, and $\pi_1(Y, y_0)$ is Abelian. Then $f_* = g_*$.*

Proof. We have $g_* = h_* \circ f_*$ where $h_*([\gamma]) = [\overline{\alpha} \cdot \gamma \cdot \alpha]$ where $\alpha(t) = F(x_0, t)$ and $f \overset{F}{\sim} g$. Now, α is a loop in Y based at y_0, so $h_*([\gamma]) = [\overline{\alpha}] \cdot [\gamma] \cdot [\alpha] = [\gamma]$ for all $[\gamma] \in \pi_1(X, x_0)$ because $\pi_{(Y, y_0)}$ is Abelian. Thus, $g_*[\gamma] = f_*[\gamma]$ holds for all $[\gamma] \in \pi_1(X, x_0)$. □

Proposition 8.10. *Given spaces X and Y, with $x_0 \in X$, $y_0 \in Y$, we have:*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. Consider the projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ defined $(x, y) \mapsto x$ and $(x, y) \mapsto y$, respectively. These induce homomorphisms $(p_X)_*$ and $(p_Y)_*$ from $\pi_1(X \times Y, (x_0, y_0))$ to $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, and so a homomorphism $h = ((p_X)_*, (p_Y)_*)$ from $\pi_1(X \times Y, (x_0, y_0))$ to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. h is an isomorphism as it has an inverse h': given loops σ at x_0 and τ at y_0, let $h'([\sigma], [\tau]) = [\gamma]$ where γ is the loop (σ, τ) at (x_0, y_0) defined by $(\sigma, \tau)(t) = (\sigma(t), \tau(t))$ for all $t \in [0, 1]$. □

Theorem 8.11. *Let X be a space which can be written as the union of two simply connected open sets U and V in such a way that $U \cap V$ is path-connected. Then X is simply connected.*

Corollary 8.12. *The sphere S^n is simply connected for each $n > 1$.*

Proof. Take two points $x \neq y \in S^n$ and let $U = S^n \setminus \{x\}$, $V = S^n \setminus \{y\}$. Then $U \cap V$ is path-connected. Moreover, U and V are homeomorphic to $\mathbb{R}^n$ and so are simply connected. Thus, S^n is simply connected. □

9. Orbit spaces, covering maps, and computing π_1

In this section, we describe some techniques for calculating the fundamental group of a topological space.

First observe that the set of homeomorphisms h from a topological space X to itself forms a group under function composition. Suppose G is a subgroup of this group. We will let X/G denote the quotient space obtained by identifying the elements of each orbit of G. In other words, let $x \sim x'$ iff there exists some $g \in G$ such that $x' = g(x)$.

Example 9.1. Take $X = \mathbb{R}$ and $G = \{h_n \mid n \in \mathbb{Z}\}$ where $h_n(x) = x + n$ for all $x \in \mathbb{R}$. Then $(G, \circ) \cong (\mathbb{Z}, +)$ since $h_n \circ h_m = h_{n+m}$ and $X/G \cong S^1$.

We say that G has a *covering space action* on X if each point $x \in X$ has a nhood U which satisfies $U \cap g(U) = \varnothing$ for all $g \in G \setminus \{e\}$. The example above is of this type.

Example 9.2. *The sphere and its quotient (projective space)*: Let $a : S^n \to S^n$ be the antipodal map. Then $G = (1_{S^n}, a) \cong \mathbb{Z}_2$ has a covering space action on S^n, and S^n/G is homeomorphic to P^n.

Example 9.3. The group $\mathbb{Z} \times \mathbb{Z}$ acts on $X = \mathbb{R}^2$ and gives the torus as the quotient space X/G. Take $g : (x, y) \mapsto (x + 1, y)$ and $h : (x, y) \mapsto (x, y + 1)$. Let $G = \langle g, h : gh = hg \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Then $\mathbb{R}^2/G \cong S^1 \times S^1$.

A group corresponding action that results in the Klein bottle K as the quotient space: let $g : (x, y) \mapsto (x + 1, y)$ and $k : (x, y) \mapsto (-x, y + 1)$. Let $G = \langle g, k : gkg = k \rangle$ is non-Abelian. For if it were, then $g = g^{-1}$ (a contradiction).

The only group action on E^2 is trivial, which follows from the fixed point theorem. Hence, cannot use E^2 for torus.

Theorem 9.4. *Let G be a group of homeomorphisms of a simply connected space X. If G has a covering space action on X, then the fundamental group $\pi_1(X/G)$ is isomorphic to G.*

Proof. Fix a point $x_0 \in X$ and given $g \in G$, join x_0 to $g(x_0)$ by a path α_g. Let $p : X \to X/G$ be the projection quotient map ($x \mapsto [x]$). Then $p \circ \alpha_g$ is a loop based at $p(x_0)$ in X/G. Let $\phi : G \to \pi_1(X/G, p(x_0))$ be defined $g \mapsto [p \circ \alpha_g]$. ϕ is well-defined since X is simply connected so any two paths from x_0 to $g(x_0)$ are homotopic relative $0, 1$. Claim: ϕ is a homomorphism (i.e., $\phi(g_1 \circ g_2) = \phi(g_1) \cdot \phi(g_2)$). Suppose $g_1, g_2 \in G$. Then $g_1 \circ \alpha_{g_2}$ is a path from $g_1(x_0)$ to $g_1 \circ g_2(x_0)$ and so $\alpha_{g_1} \cdot (g_1 \circ \alpha_{g_2})$ is a path from x_0 to $g_1 \circ g_2(x_0)$. Thus,

$$\begin{aligned} \phi(g_1 \circ g_2) &= [p \circ (\alpha_{g_1} \cdot (g_1 \circ \alpha_{g_2}))] \\ &= [p \circ \alpha_{g_1}] \cdot [p \circ (g_1 \circ \alpha_{g_2})] \\ &= \phi(g_1) \cdot \phi(g_2), \end{aligned}$$

where $[p \circ (g_1 \circ \alpha_{g_2})] = \phi(g_2)$ because $p \circ (g_1 \circ \alpha_{g_2}) = p \circ \alpha_{g_2}$ since $p \circ g_1(x) = p(x)$ for all $x \in X$. One can also show ϕ is a bijection. □

Corollary 9.5.

$$\begin{aligned} \pi_1(S^1) &\cong \mathbb{Z}, \\ \pi_1(P^n) &\cong \mathbb{Z}_2, n > 1, \textit{ and} \\ \pi_1(S^1 \times S^1) &\cong \mathbb{Z} \times \mathbb{Z}. \end{aligned}$$

For the Klein bottle, K, $\pi_1(K)$ is non-Abelian.

In particular, the sphere, the projective plane, the torus and the Klein bottle are all topologically distinct.

Proof. For S^1, take $G = \{h_n \mid n \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. $\mathbb{R}$ is simply connected and $\mathbb{R}/G \cong S^1$, so $\pi_1(S^1) \cong (\mathbb{Z}, +)$. From this, it follows that $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

For $X = P^n$, take $G = \{1, a\} \cong \mathbb{Z}_2$, which is a covering space action on S^n. Since S^n is simply connected and $X \cong S^n/G$, it follows $\pi_1(X) \cong \mathbb{Z}_2$.

For $\pi_1(K) \cong G_k$ (as above), so it is non-Abelian. □

Example 9.6.

- An example of a group action that is not a covering space action. Consider the unit sphere $X = S^2$ and the map $h : S^2 \to S^2$ defined $h(x, y, z) = h(x, y, -z)$. Then h is a homeomorphism, but it is not a covering space action on X (consider the equator) and $h \circ h = 1_X$ for $G = \{1_X, h\}$ we have X/G is homeomorphic to E^2, which is contractable and so has a trivial fundamental group. Since X is simply connected, this shows that Theorem 9.4 is not true without the requirement that G has a covering space action on X.

  Note. A simpler example is provided by $X = [-1, 1]$ and the map $h(x) = -x$ (consider 0).

- An example of a covering space action by a group G on a non-simply connected space X with $\pi_1(X/G)$ not isomorphic to G. Let $X = S^1$, the unit circle, and $G = \{1, a\}$ where a is the antipodal map. Then G has a covering space action on X, but X/G is homeomorphic to $X = S^1$ and so $\pi_1(X/G)$ is the infinite cyclic group, while G is the cyclic group of order 2.
- What is the generator of $\pi_1\big(S^1\big)$? Regard S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$. Take base point to be $z = 1$ and let $\gamma : [0,1] \to S^1$ be the loop based at 1 defined by $\gamma(s) = e^{2\pi i s}$. Notice γ winds around S^1 once, and γ^k winds around S^1 k times. Thus, $[\gamma^k] = [\gamma]^k$. It can be shown that $[\gamma]$ generates $\pi_1\big(S^1, 1\big)$.

Proposition 9.7. $X = [0,1) \times (0,1)$ *is not homeomorphic to* $Y = (0,1) \times (0,1)$.

Proof. If □

Proposition 9.8. $\mathbb{R}^3 \setminus \{(x,0,0) \mid x \in \mathbb{R}\}$ *is not simply connected.*

Proof. Let $X = \mathbb{R}^3 \setminus \{(x,0,0) \mid x \in \mathbb{R}\}$ and $Y = \{0\} \times \big(\mathbb{R}^2 \setminus \{(0,0)\}\big)$. Then $F : X \times [0,1] \to X$ defined by $F((x,y,z),t) = ((1-t)x, y, z)$ is a deformation retraction onto Y. Observe that $Y \cong \mathbb{R}^2 \setminus \{(0,0)\}$. But $H : \big(\mathbb{R}^2 \setminus \{(0,0)\}\big) \times [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined $H(x,t) = \frac{x}{(1-t)+t\|x\|}$ is a deformation retraction from $\mathbb{R}^2 \setminus \{(0,0)\}$ onto S^1. This implies that $\mathbb{R}^2 \setminus \{(0,0)\} \approx S^1$ and consequently $Y \approx S^1$. Thus, $X \approx S^1$, where S^1 is not simply connected (and therefore X is not simply connected). □

Proposition 9.9. *The compact connected* 3*-manifolds* P^3 *and* $S^1 \times S^2$ *are not homotopy equivalent.*

Proof. Observe that $\pi_1\big(P^3\big) \cong \mathbb{Z}_2$ and

$$\pi_1\big(S^1 \times S^2\big) \cong \pi_1\big(S^1\big) \times \pi_1\big(S^2\big) \cong \mathbb{Z} \times \{0\} \cong \mathbb{Z},$$

so $P^3 \not\approx S^1 \times S^2$ because $\mathbb{Z}_2 \not\cong \mathbb{Z}$. □

10. Applications of algebraic topology

Theorem 10.1. *There is no retraction from* E^2 *onto its boundary* S^1.

Proof. Suppose there was such a retraction r. Observe that

$$\underbrace{S^1 \overset{i}{\hookrightarrow} E^2 \overset{r}{\to} S^1}_{r \circ i = 1_{S^1}},$$

so we get that

$$\underbrace{\pi_1\big(S^1\big) \overset{i_*}{\to} \pi_1\big(E^2\big) \overset{r_*}{\to} \pi_1\big(S^1\big)}_{r_* \circ i_* = (r\circ i)_* = \left(1_{S^1}\right)_* = 1_{\pi_1(S^1)}}.$$

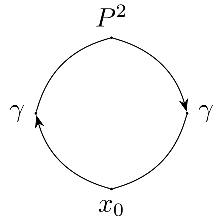
But $\pi_1\big(S^1\big) \cong \mathbb{Z}$ and $\pi_1\big(E^2\big)$ is the trivial group, so $r_* \circ i_*$ maps $\pi_1\big(S^1\big)$ to zero element of $\pi_1\big(S^1\big)$, which contradicts that $r_* \circ i_* = 1_{\pi_1(S^1)}$. □

Example 10.2. Let $X = S^2 \cup \big(E^2 \times \{0\}\big) \subset \mathbb{R}^3$. Then there is a covering space action of $\mathbb{Z}_3$ on X. X is simply connected by taking $U = X \setminus \{n\} \approx S^2$ and $V = X \setminus \{s\} \approx S^2$ (taking n as north pole and s as south pole).

Theorem 10.3. (The 'fundamental theorem of algebra'). *Any polynomial with complex coefficients that is not constant has a root in* $\mathbb{C}$.

Proof. Without loss of generality, we can take $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$ (we will derive a contradiction). For each $t \in \mathbb{R}^{\geq 0}$ defined $f_t : S^1 \to S^1$ (with $S^1 \subseteq \mathbb{C}$, $\{z \mid |z| = 1\}$) by $f_t(z) = \frac{p(tz)}{\|p(tz)\|}$. f_t is well-defined (since $p(tz) \neq 0$ for all $z \in \mathbb{C}$) and f_t is continuous. The f_0 is the constant map ($f_0(z) = \frac{p(0)}{\|p(0)\|}$ for all $z \in \mathbb{C}$) and for each t, f_t is homotopic to f_0 by the homotopy $F : S^1 \times [0,1] \to S^1$ defined $F(s,t') = f_{(1-t')t}(s)$. Then the induced homomorphism $(f_t)_* : \pi_1(S^1) \to \pi_1(S^1)$ is equal to $(f_0)_*$ which sends a generator $[\gamma]$ of $\pi_1(S^1)$ to 0 times itself. However, for sufficiently large t, f_t is homotopic to $g(z) = z^n$ and so $g_* = (f_t)_*$. But $g_* : \pi_1(S^1) \to \pi_1(S^1)$ sends the generator of $\pi_1(S^1)$ to n times itself (which is a contradiction). □

Example 10.4. What does an element of order 2 in $\pi_1(P^2) \cong \mathbb{Z}_2$ look like? P^2 as disk with antipodal points on boundary identified; take x_0 to be point on boundary.



10.1. **Homotopy and 3-dimensional manifolds.** *Knot groups.* Given a knot $k \subset \mathbb{R}^3$, its associated *knot group* is the group $\pi_1(\mathbb{R}^3 \setminus k)$. Note that equivalent knots have isomorphic groups (however the converse is not true). Thus, one way to prove that two knots are not equivalent is to compute their associated groups and show that these groups are non-isomorphic. For example, it can be shown that for the unknot (i.e., the standard embedding of S^1 in $\mathbb{R}^3$) the associated knot group is isomorphic to $\mathbb{Z}$; however, for the trefoil knot, the group is not Abelian (and so it is not isomorphic to $\mathbb{Z}$). This formally proves that the trefoil knot cannot be 'unknotted'.

10.2. **Links.** Two copies of S^1 can be embedded in $\mathbb{R}^3$ in two different ways - either 'linked' (L) or 'unlinked' (U). The former is Abelian, whereas the latter is non-Abelian.

The topological study of knots, links and related structures ('braids') has been and continues to be a rich area of research.

We end with a famous conjecture (recently proved) which states the following:

- If a compact n-manifold M is homotopy equivalent to S^n, then M is homeomorphic to S^n.

The case $n = 3$ is/was referred to as the *Poincare conjecture*. The case $n = 2$ clearly holds from the classification of surfaces. In 1961, Stephen Smale showed that the generalised Poincare conjecture holds in dimension n for all values of n greater than 4. The case where $n = 4$ was proved by Michael Freedman in 1982. The remaining case $n = 3$ (the original conjecture by Henri Poincare from around 1905) turned out to be the most difficult to decide. It is equivalent to the following statement:

- Any compact, connected 3-manifold M that is simply connected is homeomorphic to S^3.

Finally, in 2003, Russian mathematician Grigori Perelman proved that the Poincare conjecture does indeed hold in dimension $n = 3$.

11. Further applications and additional results

11.1. **Covering spaces and liftings.** In this section we assume all spaces are path connected and locally path connected (recall that a space is locally path connected if, for every point x in this space, and every nhood U of x, there is a path-connected nhood V of x contained in U).

A locally path-connected space that is not path-connected: $(0,1) \cup (1,2)$.

A path-connected space that is not locally path connected: Topologist's sine curve plus a line from origin to $(1,0)$.

Given topological spaces X and Y, and a function $p : X \to Y$ we say that p is a *covering map* if p is continuous and each $y \in Y$ has a nhood V for which $p^{-1}(V)$ is a disjoint union of open sets $\{U_\alpha\}$, and the restriction of p to U_α maps U_α homeomorphically to V (for each α). In addition, we say that the pair (X, p) is a *covering space* of Y.

Lemma 11.1. *If G has a covering space action on X, then the projection $p : X \to X/G$ is a covering map.*

Proof. Given $y \in X/G$ select a point $x \in p^{-1}(y)$ and an open nhood U of x in X such that $g(U) \cap U = \varnothing$ for all $g \neq 1_G \in G$. Now, $p : X \to X/G$ is an open map, so $V = p(U)$ is an open nhood of y. Then $\{g(U) \mid g \in G\}$ is a family of open sets $\{U_\alpha\}$ that show p is a covering map. □

Theorem 11.2. *Suppose that $f : (X, x_0) \to (Y, y_0)$ is a continuous function and $p : \left(\tilde{Y}, \tilde{y_0}\right) \to (Y, y_0)$ is a covering map. Then there exists a lifting g of f (i.e., $p \circ g = f$) if and only if $f_*(\pi_1(X, x_0)) \subseteq p_*\left(\pi_1\left(\tilde{Y}, \tilde{y_0}\right)\right)$, and this lifting is unique.*

Proof. We only provide a proof of necessity. To this end, suppose the lifting map g exists. Then

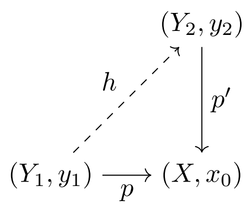
$$f_*(\pi_1(X, x_0)) = (p \circ g)_*(\pi_1(X, x_0)) = p_*(g_*(\pi_1(X, x_0))).$$

Now, $g_*(\pi_1(X, x_0)) \subseteq \pi_1\left(\tilde{Y}, \tilde{y_0}\right)$ and thus $f_*(\pi_1(X, x_0)) \subseteq p_*\left(\pi_1\left(\tilde{Y}, \tilde{y_0}\right)\right)$. □

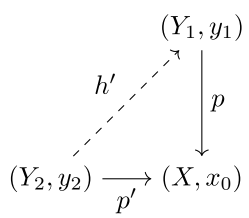
Note. When X is simply connected, a lifting g will always exist trivially (since X would have the trivial group). We will use this fact later in the proof of the Borsuk-Ulam theorem. It is also relevant as it implies that a simply connected covering space of X is 'universal' in the sense that it covers every other covering space of X.

Theorem 11.3. *If Y_1 and Y_2 are simply-connected covering spaces for X, then Y_1 and Y_2 are homeomorphic.*

Proof. We have that the lifting h exists such that



because $\pi_1(Y_1)$ is the trivial group. Reversing roles of Y_1 and Y_2 gives us another lifting map h', as shown below



again because $\pi_1(Y_2)$ is the trivial group. Trivially, the identity map on Y_1 and Y_2 are lifting maps and are equal to $h' \circ h$ and $h \circ h'$, respectively. This gives us that Y_1 and Y_2 are homeomorphic. □

Thus up to homeomorphism, if a space X has a simply connected covering space it is essentially unique (and is called the 'universal covering space' of X. The universal covering space of G_2 is similar to a 'tree' in the plane.

It can be shown that there is a covering map from the torus to the Klein bottle (which is 2-to-1).

$\mathbb{R}^2$ cannot be a covering space of S^2, since if it was, they would be homeomorphic (and yet $\mathbb{R}^2$ is not compact and S^2 is compact).

Not every space has a simply-connected covering space. An example is provided by the 'Hawaiian earring' space.

11.2. **Orthogonal transformations and maps on spheres.** An orthogonal linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ is either a rotation or a reflection depending on whether or not a matrix representing L has determinant $+1$ or -1 respectively. Note that L restricts to a map on the unit sphere S^{n-1} centred on the origin.

Lemma 11.4. *For a sphere of any dimension:*

(1) *Any rotation is homotopic to the identity map.*
(2) *Any reflection is not homotopic to the identity map.*

Consider now the *antipodal map* $a : \mathbb{R}^n \to \mathbb{R}^n$ defined by $a(x) = -x$. This orthogonal linear transformation is represented by the matrix $-I$ and so it is a rotation when n is even and a reflection when n is odd (recall that $\det(-I) = (-1)^n$). Thus we have the following corollary.

Corollary 11.5. *The antipodal map a is homotopic to the identity map on S^n if and only if n is odd.*

Theorem 11.6. *Suppose that $n \geq 2$ is even. Then for any map $f : S^n \to S^n$, there is a point $x \in S^n$ so that $f(x) = x$ or $f(x) = -x$ (the antipodal point to x).*

Proof. Suppose f satisfies $f(x) \neq x, -x$ for all $x \in S^n$. We will show that $f \sim 1_{S^n}$ and $f \sim a_{S^n}$ and so by transitivity $1_{S^n} \sim a_{S^n}$ which implies that n is odd. Consider the great circle path $\gamma_x(s) : [0,1] \to S^n$ for x to $f(x)$ which is well-defined since $f(x) \neq -x$. Let $F : S^n \times [0,1] \to S^n$ be the map $(x,t) \mapsto \gamma_x(t)$. Then $F(x,0) = x$ and $F(x,1) = f(x)$ so $f \sim 1_{S^n}$. A similar argument shows that $f \sim a_{S^n}$, and so it follows that n is odd (as desired). □

This result is closely related to (and implies) the famous 'hairy ball theorem': at any moment there is a place on the surface of the earth where the wind is not blowing! More precisely, it says that any 'vector field' on the surface of a sphere must vanish at one point at least.

Corollary 11.7. *Let $v : S^2 \to \mathbb{R}^3$ be a continuous function for which $v(s)$ lies in the tangent plane to s for each $s \in S^2$. Then $v(s) = 0$ for some point $s \in S^2$.*

Note. The analogue of this corollary does not hold for a torus.

Corollary 11.8. *If $L : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, and n is odd, then L maps a line through the origin onto itself.*

Proof. Write $n = 2k+1$ and let $S^{2k} \subseteq \mathbb{R}^{2k+1}$ be the unit sphere centred on 0. Consider the map $\varphi : S^{2k} \to S^{2k}$ defined $x \mapsto \frac{L(x)}{\|L(x)\|}$. Since L is linear and invertible, φ is well-defined and continuous. By Theorem 11.6, there exists a point $x_0 \in S^{2k}$ such that $\varphi(x_0) \in \{x_0, -x_0\}$. In either case, $L(x_0) = cx_0$ for some constant c and so the line ℓ through 0 and x_0, $\ell = (sx_0 : s \in \mathbb{R})$, satisfies $L(sx_0) = (cs)x_0 \in \ell$. Thus, L maps ℓ to itself. □

11.3. **The Borsuk-Ulam (BU) theorem.**

Theorem 11.9. (BU theorem (Version 1)). *Let $n \geq 1$. There is no map $f : S^n \to S^{n-1}$ that commutes with the antipodal map (i.e., satisfies $f(-x) = -f(x)$ for all $x \in S^n$).*

Proof. We prove for cases $n \leq 2$. The case $n = 1$ is clear ($S^0 = \{-1, 1\}$ is disconnected, S^1 is connected). Suppose $n = 2$ and $f : S^2 \to S^1$ satisfies $f(-x) = -f(x)$ for all $x \in S^2$ (we derive a contradiction). Since $(f')_*\big(\pi_1\big(S^1\big)\big)$ is the identity element of $\pi_1\big(S^1\big)$, there exists a lifting of f', g, to $\mathbb{R}$ so $f' = p \circ g$.

But $g : S^1 \to \mathbb{R}$ has the property that there exists x_0 such that $g(x_0) = g(-x_0)$ and so $f(-x_0) = p \circ g(-x_0) = p \circ g(x_0) = f(x_0)$ which contradicts the assumption that $f(-x) = -f(x)$ (for all x). □

Theorem 11.10. (BU theorem (Version 2)). *Let $n \geq 1$. If $f : S^n \to \mathbb{R}^n$ is continuous, then f identifies a pair of antipodal points (i.e., there exists some $x \in S^n$ for which $f(x) = f(-x)$).*

Proof. Suppose $f : S^n \to \mathbb{R}^n$ is continuous, and $f(x) \neq f(-x)$ for all $x \in S^n$. Let $g : S^n \to \mathbb{R}^n$ be defined $g(x) = f(x) - f(-x)$, which satisfies $g(-x) = -g(x)$ (odd function) and $g(x) \neq 0$ for all $x \in S^n$. Let $h : S^n \to S^{n-1}$ be $x \mapsto \frac{g(x)}{\|x\|}$, which is well-defined, continuous, and $h(-x) = -h(x)$, which contradicts the previous result. □

This theorem has a nice interpretation in the case where $n = 2$. Take any two variables that vary continuously over the surface of the earth (e.g. temperature and air pressure). Then at any given moment, there is always a spot on the surface of the earth where both variables are identical to the values on the antipodal point on the earth. An immediate corollary is that S^n cannot be embedded in $\mathbb{R}^n$.

The BU theorem has many applications (including in combinatorics). One famous application is the so-called 'ham sandwich theorem': For any compact sets $A_1, \ldots, A_n$ in $\mathbb{R}^n$, we can always find a hyperplane dividing each of them into two subsets of equal measure.

Another corollary of the BU theorem is the following (Lusternik-Shnirelman theorem).

Corollary 11.11. *If S^n is covered by $n+1$ closed sets, then one of the sets contains a pair of antipodal points.*

Proof. We give a proof for case $n = 2$. Call the three closed sets A_1, A_2, A_3. Let $d_i : S^2 \to \mathbb{R}$ measure distance to A_i, that is, $d_i(x) = \inf_{y \in A_i} |x - y|$. Then the map from $\varphi : S^2 \to \mathbb{R}^2$ defined $x \mapsto (d_1(x), d_2(x))$. By the BU Theorem (Version 2) there exists a point $x \in S^2$ such that $\varphi(x) = \varphi(-x)$ (i.e., $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$). If either $d_1(x)$ or $d_2(x)$ is zero, then x and $-x$ both lie in A_1 or A_2, since these sets are closed. On the other hand, if the distance from x and $-x$ to A_1 and A_2 are strictly positive, then x and $-x$ lie in A_3. □